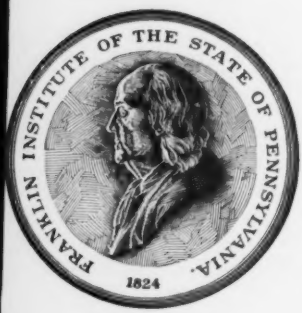
SEPTEMBER 1961

Journal
of
The Franklin Institute

1826



1961

Statistical Design of Discrete-Data Control Systems Subject to Power Limitation

JULIUS T. TOU AND K. S. PRASANNA KUMAR

Stresses in Cylindrically Symmetric Membranes Reinforced with Extensible Cords

W. F. AMES

Analytical Solutions for Rotary Matrix Wire Screen Heat Exchangers

W. H. T. LOH

The Minimal Time Regulator Problem for Linear Sampled-Data Systems: General Theory

C. A. DESOER AND J. WING

VOL. 272 NO. 3

AWARDS BY THE INSTITUTE

The Franklin Medal (1914—Gold Medal).—This medal is awarded annually to those workers in physical science or technology, without regard to country, whose efforts, in the opinion of the Institute, acting through its Committee on Science and the Arts, have done most to advance a knowledge of physical science or its applications.

The Elliott Cresson Medal (1848—Gold Medal).—This medal is awarded for discovery or original research, adding to the sum of human knowledge, irrespective of commercial value; leading and practical utilizations of discovery; and invention, methods or products embodying substantial elements of leadership in their respective classes, or unusual skill or perfection in workmanship.

The Howard N. Potts Medal (1906—Gold Medal).—This medal is awarded for distinguished work in science or the arts; important development of previous basic discoveries, inventions or products of superior excellence or utilizing important principles.

The John Price Wetherill Medal (1925—Silver Medal).—This medal is awarded for discovery or invention in the physical sciences or for new and important combinations of principles or methods already known.

The Edward Longstreth Medal (1890—Silver Medal).—This medal is awarded for inventions of high order and for particularly meritorious improvements and developments in machines and mechanical processes. In the event of an accumulation of the fund for medals beyond the sum of one hundred dollars, it is competent for the Committee on Science and the Arts to offer from such surplus a money premium for some special work on any mechanical or scientific subject that is considered of sufficient importance.

The Louis E. Levy Medal (1923—Gold Medal).—This medal is awarded to the author of a paper of especial merit, published in the JOURNAL OF THE FRANKLIN INSTITUTE, preference being given to one describing the author's experimental and theoretical researches in a subject of fundamental importance.

The George R. Henderson Medal (1924—Gold Medal).—This medal is awarded for meritorious inventions or discoveries in the field of Railway Engineering.

The Walton Clark Medal (1926—Gold Medal).—This medal is awarded to the "author of the most notable advance in knowledge or improvement in apparatus, or in method concerning the science or the art of gas manufacture or distribution or utilization in the production of illumination, or of heat, or of power."

The Frank P. Brown Medal (1938—Silver Medal).—This medal is awarded to inventors for discoveries and inventions involving meritorious improvements in the building and allied industries.

The Newcomen Medal (1943—Gold Medal).—This medal is awarded, not oftener than once in three years, for achievement in the field of Steam.

The Francis J. Clamer Medal (1943—Silver Medal).—This medal is awarded at least once in five years for meritorious achievement in the field of Metallurgy.

The Stuart Ballantine Medal (1946—Gold Medal).—This medal is to be awarded in recognition of outstanding achievement in the fields of Communication and Reconnaissance which employ electromagnetic radiation.

The Boyden Premium (1859).—This premium is awarded not oftener than once in five years to any resident of North America who has recently made a notable experimental determination of the speed, in free space, of radiation in any region of the entire spectrum.

The William M. Vermilye Medal (1937—Bronze Medal).—This medal is awarded not oftener than biennially in recognition of outstanding contribution in the field of Industrial Management.

The Certificate of Merit (1882).—A Certificate of Merit is awarded to persons adjudged worthy thereof for meritorious inventions, discoveries or improvements in physical processes or devices.

For further information relating to these awards apply to the Secretary, Committee on Science and the Arts

INDEX TO ADVERTISERS

	PAGE
Atlas Photo Engraving Co.	viii
Bell Telephone Laboratories	vi
Biddle, James G., Co.	x
Buten, M., & Sons	iv
Coleman, W. B., & Co.	ii
Damon & Foster	ii
Davidson, P. L.	ii
Franklin Institute, Computing Center	Fourth Cover Page
Franklin Institute, Laboratories	Third Cover Page
Harris-Dechant Associates	ii
Herbach & Rademan	viii
Hess & Barker	viii
Lancaster Press, Inc.	v
Leopold, Charles S.	ii
Levy & Co., Max	iv
Shanahan & Co.	viii
Thomas Co., Arthur H.	iv
Yarnall-Waring Co.	ix
Young, George, Co.	viii

Published monthly by The Franklin Institute of the State of Pennsylvania, Benjamin Franklin Parkway at Twentieth St., Philadelphia 3, Pennsylvania. Printed by Lancaster Press, Inc., Prince and Lemon Sts., Lancaster, Pennsylvania.

Subscription rates: Domestic, $10.00 a year; Foreign, $11.00 a year. Single current issues, $1.00. Publication of reprints for sale was discontinued in July, 1959. All inquiries regarding subscriptions, change of address, and sale of back issues, should be addressed to the Philadelphia office.

Indexes to the semi-annual volumes of the JOURNAL are published in the June and December numbers.

Copyright © 1961, by The Franklin Institute of the State of Pennsylvania.

Second-class postage paid at Lancaster, Pa.

Consulting Engineers

HARRIS-DECHANT ASSOCIATES

FREDERICK H. DECHANT
Consulting Engineer

123 So. Broad St.
Philadelphia 9, Pa.

P. L. DAVIDSON
Consulting Engineer

◆

Philadelphia, Pa. and
Greensboro, N. C.

DAMON & FOSTER

Consulting Engineers
Surveyors

CHESTER PIKE & HIGH ST.
SHARON HILL, PA.

CHARLES S. LEOPOLD, INC.
ENGINEERS

◆

215 SOUTH BROAD ST.
PHILA. 7, PA.

W. B. COLEMAN & CO.
Metallurgists—Chemists—Engineers

Spectrographic Analysis
Chemical and Physical Testing
Metallurgical Investigations
Boiler Water Conditioning
Consultation Service

9th and Rising Sun Ave., Philadelphia 40, Pa.

Member: American Council of Independent Laboratories

THE FRANKLIN INSTITUTE

OFFICERS

President.....WYNN LAURENCE LEPAGE

Executive Vice President.....J. G. RICHARD HECKSCHER
Secretary.....W. F. JACKSON, JR.
Treasurer.....C. M. WATERBURY
Director of Laboratories.....

Assistant Secretaries.....C. M. WATERBURY, WALTER A. R. PERTUCH
Assistant Treasurer.....JOHN J. JACKSON

BOARD OF MANAGERS

Term expires in 1962
HENRY B. ALLEN
G. H. CLAMER
A. FELIX DUPONT, JR.
C. L. JORDAN
LIONEL F. LEVY
R. G. RINCLIFFE
JAMES H. ROBINS
EDWARD H. SMOKER
PHILIP H. WARD, JR.
HARLESTON R. WOOD

Term expires in 1963
BRANDON BARRINGER
J. FRANK COX
JOSEPH GRAY JACKSON
RICHARD T. NALLE
S. WYMAN ROLPH
JOHN RUSSELL, JR.
W. MAXWELL SCOTT, JR.
I. MELVILLE STEIN

Term expires in 1964
HENRY B. BRYANS
HENRY M. CHANCE, II
FRANCIS J. CHESTERMAN
JAMES CREESE
RUPEN EKSERGIAN
JOSEPH A. FISHER
MORTON GIBBONS-NEFF
GAYLORD P. HARNWELL
WYNN LAURENCE LEPAGE
EMERY W. LOOMIS

EXECUTIVE COMMITTEE OF THE BOARD OF MANAGERS

RICHARD T. NALLE, *Chairman*
HENRY B. BRYANS — HENRY M. CHANCE, II — WYNN LAURENCE LEPAGE
JOSEPH A. FISHER

STANDING COMMITTEES OF THE BOARD OF MANAGERS

Nominating Committee.....RICHARD T. NALLE, *Chairman*
Budget Committee.....HENRY B. BRYANS, *Chairman*
Personnel Policy Committee.....HARLESTON R. WOOD, *Chairman*
Buildings and Grounds Committee.....W. MAXWELL SCOTT, JR., *Chairman*
Education Committee.....JOSEPH GRAY JACKSON, *Chairman*
Research and Development Committee.....G. H. CLAMER, *Honorary Chairman*; EMERY W. LOOMIS, *Chairman*
Bartol Research Foundation Committee.....G. H. CLAMER, *Honorary Chairman*; GAYLORD P. HARNWELL, *Chairman*
Biochemical Research Foundation Committee.....A. FELIX DUPONT, JR., *Chairman*
Educational Radio and Television Committee.....LIONEL F. LEVY, *Chairman*

GENERAL COMMITTEES OF THE INSTITUTE

Committee on Science and the Arts.....FRANCIS G. TATNALL, *Chairman*
Membership Committee.....MORTON GIBBONS-NEFF, *Chairman*
Investment Committee.....BRANDON BARRINGER, *Chairman*
Endowment Committee.....MORTON GIBBONS-NEFF, *Chairman*

ADVISORY COMMITTEES OF THE INSTITUTE

Museum and Memorial Committee.....I. MELVILLE STEIN, *Chairman*
Library Committee.....PENROSE R. HOOPES, *Chairman*
Meetings Committee.....LESLIE J. WOODS, *Chairman*
Public Relations Committee.....C. L. JORDAN, *Chairman*
Publications Committee.....LIONEL F. LEVY, *Chairman*
Laboratories Committee.....RUPEN EKSERGIAN, *Chairman*

HOSTESS COMMITTEE OF THE INSTITUTE

MISS MARTHA B. NEWKIRK, *Chairman*

PRECISION RULINGS ON GLASS

Scales · Grids · Reticles
Halftone Screens

MAX LEVY & CO. · Wayne Ave. & Berkley St. Phila., Pa.

EVERYTHING IN PAINTS

BUTEN'S

PAINT STORES

Phila., Chester, Reading, Upper Darby, Bryn Mawr, Darby, Doylestown
Camden, N. J. — Wilmington, Del.

ARTHUR H. THOMAS CO.

Laboratory Apparatus and Reagents

Selected for Chemistry and Biology

Adequate stocks of 16,000 Apparatus items and 6,000 Reagent items carried in our Philadelphia warehouse for immediate shipment.

VINE STREET AT THIRD **PHILADELPHIA 5, PA.**

A COMPLETE PRINTING SERVICE

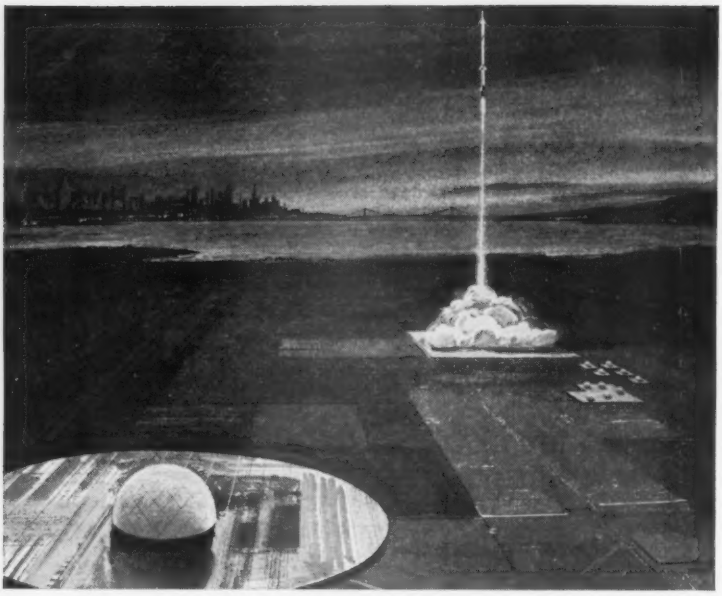
GOOD PRINTING does not just happen; it is the result of careful planning. The knowledge of our craftsmen, who for many years have been handling details of composition, printing and binding, is at your disposal. For over seventy-five years we have been printers of scientific and technical journals, books, theses, dissertations and works in foreign languages. Consult us about your next job.

. PRINTERS OF .
JOURNAL OF THE
FRANKLIN INSTITUTE

LANCASTER PRESS, Inc.

PRINTERS • BINDERS • ELECTROTYPERS

ESTABLISHED 1877 LANCASTER, PA.



More than 450,000 pounds of thrust lifts the U. S. Army's Nike Zeus missile skyward in a cloud of vapor. The Nike Zeus missile being developed for the project by the Douglas Aircraft Company will be designed to intercept ballistic missiles traveling over 15,000 miles per hour, and destroy them at a safe distance from the defended area.

How do you stop an ICBM?

How do you detect, track, intercept—and destroy in minutes—an ICBM that is moving through outer space ten times faster than a bullet?

Bell Telephone Laboratories may have designed the answer: Nike Zeus, a fully automated system designed to intercept and destroy all types of ballistic missiles—not only ICBM's but also IRBM's launched from land, sea or air. The system is now under development for the Army Ordnance Missile Command.

Radically new radar techniques are being developed for Nike Zeus. There will be an acquisition radar designed to detect the invading missile at great distances. And a discrimination radar designed to distinguish actual warheads from harmless decoys that may be included to confuse our defenses.

The system tracks the ICBM or IRBM, then launches and tracks the Nike Zeus missile and automatically steers it all the way to intercept the target. The entire engagement, from detection to destruction, would take place within minutes and would span hundreds of miles.

Under a prime Army Ordnance contract with the Western Electric Company, Bell Laboratories is charged with the development of the entire Nike Zeus system, with assistance from many subcontractors. It is another example of the cooperation between Bell Laboratories and Western Electric for the defense of America.

BELL TELEPHONE LABORATORIES

World center of communications research and development

Journal
of
The Franklin Institute

MANAGING EDITOR, NANCY S. GLENN

Editor Emeritus, HENRY B. ALLEN

Associate Editors:

CHARLES B. BAZZONI
WILLIAM H. BOGHOSIAN
CHARLES L. CRITCHFIELD
RUPEN EKSERGIAN
VIRGIL MORING FAIRES
PAUL D. FOOTE
DUNSTAN GRAHAM
WALTER C. JOHNSON
Y. H. KU
ROBERT S. LIVINGSTON
F. J. MURRAY
C. J. STAUD
W. F. G. SWANN
HUGH S. TAYLOR
MERLE A. TUVE

Committee on Publications:

LIONEL F. LEVY, *Chairman*
HENRY B. ALLEN
CLARENCE L. JORDAN
PHILIP H. WARD, JR.

Vol. 272 **SEPTEMBER, 1961** **No. 3**

CONTENTS

Statistical Design of Discrete-Data Control Systems Subject to Power Limitation ..... JULIUS T. TOU AND K. S. PRASANNA KUMAR 171

Stresses in Cylindrically Symmetric Membranes Reinforced with Extensible Cords ..... W. F. AMES 185

Analytical Solutions for Rotary Matrix Wire Screen Heat Exchangers ..... W. H. T. LOH 191

The Minimal Time Regulator Problem for Linear Sampled-Data Systems: General Theory ..... C. A. DESOER AND J. WING 208

National Bureau of Standards News 229

The Franklin Institute
New Books in The Franklin Institute Library 232

Book Reviews 234

Book Notes 238

Publications Received 243

Current Topics 244

Commercial Stationery
Loose Leaf—Blank Books
Filing Equipment
Office Supplies

SHANAHAN & CO.
22 S. 18 St. Ri 6-0333

WHERE QUALITY OF REPRODUCTION IS ESSENTIAL ... BUT ECONOMY IS IMPORTANT ...
Atlas
IS THE ANSWER
ATLAS PHOTO ENGRAVING CO. 1208 CHERRY ST. PHILA., 7, PA.

OFF THE SHELF DELIVERY
on thousands & thousands of
INDUSTRIAL ELECTRONIC COMPONENTS
HERBACH & RADEMAN, INC.
1204 ARCH STREET PHILADELPHIA 7, PA.
Distributors of the Finest in Electronic Equipment and Related Products for Research * Laboratory * Industry. Since 1934.

GEORGE YOUNG COMPANY
RIGGING AND HAULING
CONTRACTORS
S. W. COR. 20th & OREGON AVE.
PHILADELPHIA 45, PA. HO 7-5315

60 YEARS OF EXPERIENCE
ENGINEERING MACHINISTS
DESIGN • MANUFACTURE • REPAIRS
HESS & BARKER
930 Washington Ave. • HOward 7-1121 • Phila. 47, Pa.

YARWAY
STEAM PLANT EQUIPMENT
OVER 50 YEARS A PARTNER
IN THE PROGRESS OF
STEAM ENGINEERING
Yarnall-Waring Company • Philadelphia 18

BIDDLE
Instruments

and miscellaneous laboratory equipment cover a wide field of electrical, mechanical and scientific testing, measuring and experimental work.

For a complete file of all our products
write for Bulletin 19-53

JAMES G. BIDDLE CO.
Electrical & Scientific Instruments
1316 Arch Street—Philadelphia 7, Pa.

Journal
of
The Franklin Institute
Devoted to Science and the Mechanic Arts

Vol. 272 SEPTEMBER, 1961 No. 3

STATISTICAL DESIGN OF DISCRETE-DATA CONTROL SYSTEMS SUBJECT TO POWER LIMITATION

BY

JULIUS T. TOU[1] AND K. S. PRASANNA KUMAR[1]

SYNOPSIS

A solution is presented for the analytical design of a digital controller in accordance with a specified optimum criterion. This approach makes use of the z-transform and the modified z-transform techniques. The analytic design procedure for discrete-data control systems subject to power limitation is developed. Attention is centered upon stationary random functions because of their frequent occurrence in control systems and their being amenable to mathematical treatment. In this paper, physical realizability and system constraints are considered and general design equations are derived.

INTRODUCTION

Conventional design techniques where the system inputs are assumed to be deterministic in nature may fail to yield satisfactory results. This is due to the fact that the designer presupposes the nature of the input as either sinusoids or simple aperiodic functions. In order to broaden the design theory, stochastic input signals should be considered (**1**, **8**, **9**).[2]

The analysis and synthesis of discrete-data control systems has been a matter of considerable interest for well over ten years (**1**–**7**). Several papers have appeared on the statistical design of digital and sampled-data control systems. These papers (**1**–**3**) have formulated systematic procedures for the design of discrete-data systems satisfying a minimum mean-square-error criterion. This paper extends the concepts and ideas of the author's previous work (**1**–**3**) to the design of a discrete-data control system such that the mean-square-error is mini-

[1] The Information Technology Laboratory, the Technological Institute, Northwestern University, Evanston, Ill.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the JOURNAL.)

mized while at the same time the power of the control signal is constrained. This paper may be considered as the companion paper to reference 3 which appeared in this JOURNAL.

Power constraint is a very important aspect in control-system design. Without considering power limitations, the design may lead to a system with excessively large power dissipation in the components (8, 9), which is certainly undesirable and impractical. This paper is concerned with this important design factor. Optimum pulse-transfer functions for the digital controller are derived under physical realizability conditions. The input signals and noise are assumed to be stationary random functions.

OPTIMUM DIGITAL CONTROLLER FOR MINIMUM MEAN-SQUARE SAMPLED ERROR

The optimum design problem is solved with reference to Fig. 1. The mean-square sampled error of the system is to be minimized, meanwhile the mean-square value of the control signal $u(nT)$ is constrained. Now, let $G_o(z)$ be the over-all pulse-transfer function of the

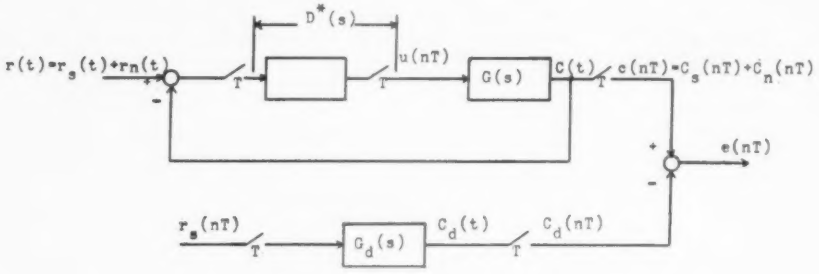


FIG. 1.

system which is to be optimized in the above sense. $G_d(z)$ is the desired over-all pulse-transfer function. The sampled input and output of the system are:

$$r(nT) = r_s(nT) + r_n(nT) \tag{1}$$

and

$$c(nT) = c_s(nT) + c_n(nT), \tag{2}$$

respectively.

The desired output sequence is $c_d(nT)$. The error sequence of the system is given by

$$e(nT) = c(nT) - c_d(nT). \tag{3}$$

The mean-square sampled error is given by

$$\overline{e^2(nT)} = \frac{1}{2\pi j} \oint_\Gamma \phi_{ee}(z) z^{-1} dz, \tag{4}$$

in which

$$\begin{aligned}\phi_{ee}(z) = {} & [G_o(z) - G_d(z)][G_o(z^{-1}) - G_d(z^{-1})]\phi_{r_s r_s}(z) \\ & + G_o(z)[G_o(z^{-1}) - G_d(z^{-1})]\phi_{r_s r_n}(z) \\ & + G_o(z^{-1})[G_o(z) - G_d(z)]\phi_{r_n r_s}(z) + G_o(z)G_o(z^{-1})\phi_{r_n r_n}(z). \end{aligned} \tag{5}$$

The mean-square value of the signal $u(nT)$ is given by

$$\overline{u^2(nT)} = \frac{1}{2\pi j}\oint_\Gamma \phi_{uu}(z)z^{-1}dz \tag{6}$$

in which

$$\phi_{uu}(z) = W(z)W(z^{-1})[\phi_{r_s r_s}(z) + \phi_{r_n r_n}(z) + \phi_{r_s r_n}(z) + \phi_{r_n r_s}(z)]. \tag{7}$$

Referring to Fig. 1,

$$G_o(z) = W(z)G(z) \tag{8}$$

where

$$W(z) = \frac{D(z)}{1 + D(z)G(z)} \tag{9}$$

and, $G(z)$ has its poles and zeros inside the unit circle. Substituting (8) into (5) yields

$$\begin{aligned}\phi_{ee}(z) = {} & [W(z)G(z) - G_d(z)][W(z^{-1})G(z^{-1}) - G_d(z^{-1})]\phi_{r_s r_s}(z) \\ & + W(z)G(z)[W(z^{-1})G(z^{-1}) - G_d(z^{-1})]\phi_{r_s r_n}(z) \\ & + W(z^{-1})G(z^{-1})[W(z)G(z) - G_d(z)]\phi_{r_n r_s}(z) \\ & + W(z)W(z^{-1})G(z)G(z^{-1})\phi_{r_n r_n}(z). \end{aligned} \tag{10}$$

The design objective is to minimize (4) subject to the constraint that (6) is less than or equal to a specified value. $\overline{u^2(nT)}$ may represent the power input to a component of the system and it may be necessary to constrain it for protection of the part concerned.

The minimization problem is solved as follows. The functional

$$F = \overline{e^2(nT)} + \lambda\overline{u^2(nT)} \tag{11}$$

is formed where

$$\lambda = \text{the Lagrange multiplier.} \tag{12}$$

Substituting (4) and (6) into (11) yields

$$F = \frac{1}{2\pi j}\oint_\Gamma \phi_{ee}(z)z^{-1}dz + \frac{\lambda}{2\pi j}\oint_\Gamma \phi_{uu}(z)z^{-1}dz. \tag{13}$$

Following the method of reference 1, the pulse-transfer function $W(z)$ is given a small variation $\gamma\eta(z)$, and $W(z^{-1})$ is given a variation $\gamma\eta(z^{-1})$. As a result, F undergoes a variation δF. $\eta(z)$ is an arbitrary function of z and γ is a parameter. $W(z)$ is replaced by its neighboring function $W(z) + \gamma\eta(z)$, and $W(z^{-1})$ by its neighboring function $W(z^{-1}) + \gamma\eta(z^{-1})$. The functional F becomes $F + \delta F$. Then, F is a minimum, if

$$\left.\frac{d[F + \delta F]}{d\gamma}\right|_{\gamma=0} = 0. \tag{14}$$

Following the operations involved in (14) and simplifying leads to

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\{W(z^{-1})[\lambda + G(z)G(z^{-1})]\phi(z) - G(z)G_d(z^{-1})\phi_1(z)\}z^{-1}dz + \frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\{W(z)[\lambda + G(z)G(z^{-1})]\phi(z) - G(z^{-1})G_d(z)\phi_2(z)\}z^{-1}dz = 0 \tag{15}$$

where

$$\phi(z) \triangleq \phi_{r_g r_g}(z) + \phi_{r_g r_n}(z) + \phi_{r_n r_g}(z) + \phi_{r_n r_n}(z) \tag{16}$$

$$\phi_1(z) \triangleq \phi_{r_g r_g}(z) + \phi_{r_g r_n}(z) \tag{17}$$

$$\phi_2(z) \triangleq \phi_{r_g r_g}(z) + \phi_{r_n r_g}(z). \tag{18}$$

The next step is to factor $\phi(z)$ and $[\lambda + G(z)G(z^{-1})]$ into two parts:

$$\phi(z) \triangleq \phi^+(z)\phi^-(z) \tag{19}$$

$$[\lambda + G(z)G(z^{-1})] \triangleq B(z) = B^+(z)B^-(z), \tag{20}$$

where the first factors of (19) and (20) contain zeros and poles inside the unit circle and the second factors contain zeros and poles outside the unit circle. From (15), (19) and (20) results

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\phi^+(z)B^+(z)\left\{W(z^{-1})\phi^-(z)B^-(z) - \frac{G(z)G_d(z^{-1})\phi_1(z)}{\phi^+(z)B^+(z)}\right\}z^{-1}dz + \frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\phi^-(z)B^-(z)\left\{W(z)\phi^+(z)B^+(z) - \frac{G(z^{-1})G_d(z)\phi_2(z)}{\phi^-(z)B^-(z)}\right\}z^{-1}dz = 0. \tag{21}$$

For physical realizability, $W(z)$ and $\eta(z)$ can have no poles outside the unit circle and $W(z^{-1})$ and $\eta(z^{-1})$ can have no poles inside the unit circle. Thus, the function $\eta(z)\phi^{+}(z)B^{+}(z)$ contains poles inside the unit circle only, and the function $\eta(z^{-1})\phi^{-}(z)B^{-}(z)$ contains poles outside the unit circle only. Consequently, (21) may be written as

$$\frac{1}{2\pi j}\oint_{\Gamma}\eta(z)\phi^{+}(z)B^{+}(z)\left[W(z^{-1})\phi^{-}(z)B^{-}(z)-\left\{\frac{G(z)G_d(z^{-1})\phi_1(z)}{\phi^{+}(z)B^{+}(z)}\right\}_{-}\right]z^{-1}dz$$

$$+\frac{1}{2\pi j}\oint_{\Gamma}\eta(z^{-1})\phi^{-}(z)B^{-}(z)\left[W(z)\phi^{+}(z)B^{+}(z)-\left\{\frac{G(z^{-1})G_d(z)\phi_2(z)}{\phi^{-}(z)B^{-}(z)}\right\}_{+}\right]z^{-1}dz=0 \quad (22)$$

in which the symbol $\{\ \}_{+}$ means the operation of picking the part of a function of z with poles inside the unit circle, and the symbol $\{\ \}_{-}$ means the operation of picking the part of a function of z with poles outside the unit circle. For instance, in $F(z)=\{F(z)\}_{+}+\{F(z)\}_{-}$, the terms with poles inside the unit circle are grouped in the $(+)$ part, and the terms with poles outside the unit circle are collected in the $(-)$ part.

If $W(z)$ is indeed the optimum transfer function, (22) must be satisfied for arbitrary $\eta(z)$. Thus, the optimum $W(z)$ is given by

$$W(z)=\frac{\left\{\dfrac{G(z^{-1})G_d(z)\phi_2(z)}{\phi^{-}(z)B^{-}(z)}\right\}_{+}}{\phi^{+}(z)B^{+}(z)}. \quad (23)$$

The pulse-transfer function $D(z)$ is determined from

$$D(z)=\frac{W(z)}{1-W(z)G(z)}. \quad (24)$$

It is noted that (23) and (24) contain λ as a parameter. The value of λ is adjusted so as to satisfy the constraint on $\overline{u^2(nT)}$, namely,

$$\overline{u^2(nT)}\leq\sigma^2. \quad (25)$$

In practice, the adjustment of λ may entail a graphical procedure if the constraint equation cannot be solved explicitly for λ. When once the value of λ is fixed, (23) and (24) yield the optimum pulse-transfer functions. From these and (4), the mean-square sample error can be calculated.

Example

$$\text{Given: } G(s) = \frac{1 - \epsilon^{-TS}}{s(1+s)}; \; G_d(s) = 1; \; T = 0.2 \text{ sec.}$$

$$\phi_{r_s r_s}(\omega) = \frac{1}{1+\omega^2}; \qquad \phi_{r_n r_n}(\omega) = 0.1$$

$$\phi_{r_n r_s}(\omega) = \phi_{r_s r_n}(\omega) = 0$$

$$\overline{u^2(nT)} \leq 2.0.$$

Determine the transfer function of the pulsed-data compensator required to minimize the mean-square sampled error. The following z-transforms follow.

$$G(z) = \frac{0.181}{(z - 0.819)}; \qquad G(z^{-1}) = \frac{-0.221z}{(z - 1.221)}$$

$$\phi_{r_s r_s}(z) = \frac{-0.2013z}{(z - 0.819)(z - 1.221)}; \qquad \phi_{r_n r_n}(z) = 0.1$$

$$\phi(z) = \left[0.316\,\frac{(z - 0.261)}{(z - 0.819)}\right]\left[0.316\,\frac{(z - 3.791)}{(z - 1.221)}\right].$$

From (20),

$$B(z) = \frac{(z-a)(z-b)}{(z - 0.819)(z - 1.221)}$$

where

$$a = \tfrac{1}{2}\left\{\left(2.04 + \frac{0.04}{\lambda}\right) - \sqrt{\left(2.04 + \frac{0.04}{\lambda}\right)^2 - 4}\right\}$$

and

$$b = \tfrac{1}{2}\left\{\left(2.04 + \frac{0.04}{\lambda}\right) + \sqrt{\left(2.04 + \frac{0.04}{\lambda}\right)^2 - 4}\right\}$$

when $\lambda > 0$, $a < 1$ and $b > 1$ and hence

$$B^+(z) = \frac{(z-a)}{(z - 0.819)} \qquad \text{and} \qquad B^-(z) = \frac{(z-b)}{(z - 1.221)}.$$

Using (23),

$$W(z) = \frac{0.1(z - 0.819)}{\lambda(b - 0.819)(z-a)(z - 0.261)}.$$

From (6),

$$\overline{u^2(nT)} = \frac{0.00314(3.791 - a)}{a\lambda^2(b - 0.819)^2\left(\frac{1}{a} - a\right)(3.83 - a)}.$$

Actually a curve of $\overline{u^2(nT)}$ v/s $\overline{e^2(nT)}$ with λ as the parameter should be plotted and the optimum λ selected therefrom. However, if $\lambda = 0.01$, $\overline{u^2(nT)} = 1.25$ which satisfies the specification $\overline{u^2(nT)} \leq 2.0$. Thus,

$$W(z) = \frac{1.98(z^{-1} - 0.819z^{-2})}{(1 - 0.437z^{-1} + 0.044z^{-2})}.$$

It follows from (9) that

$$D(z) = \frac{1.98\ (z^{-1} - 0.819z^{-2})}{(1 - 0.437z^{-1} - 0.314z^{-2})}.$$

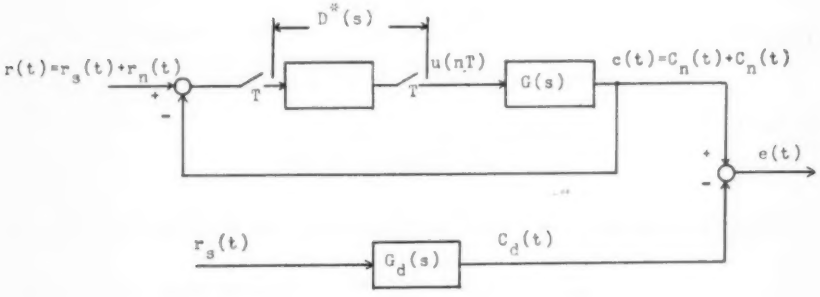


FIG. 2.

OPTIMUM DIGITAL CONTROLLER FOR MINIMUM MEAN-SQUARE ERROR

With reference to Fig. 2,

$$\begin{aligned} e(t) &= c(t) - c_d(t) \\ &= c_s(t) + c_n(t) - c_d(t). \end{aligned} \tag{26}$$

In terms of sampled values,

$$e(nT, m) = c_s(nT, m) + c_n(nT, m) - c_d(nT, m) \tag{27}$$

where n is an integer and m lies between 0 and 1. For a given value of m, the mean-square value of the error samples is

$$\overline{e^2(nT, m)} = \overline{c_d^2(nT, m)} + \overline{c_s^2(nT, m)} + \overline{c_n^2(nT, m)} + \overline{c_s(nT, m)c_n(nT, m)} + \overline{c_n(nT, m)c_s(nT, m)} - \overline{c_d(nT, m)c_s(nT, m)} - \overline{c_s(nT, m)c_d(nT, m)} - \overline{c_d(nT, m)c_n(nT, m)} - \overline{c_n(nT, m)c_d(nT, m)}. \quad (28)$$

Equation 28 reduces to

$$\overline{e^2(nT, m)} = \frac{1}{2\pi j}\oint_\Gamma \phi_{ee}(z, m)z^{-1}dz \quad (29)$$

where

$$\phi_{ee}(z, m) = \phi_{c_dc_d}(z, m) + \phi_{c_sc_s}(z, m) + \phi_{c_nc_n}(z, m) + \phi_{c_sc_n}(z, m) + \phi_{c_nc_s}(z, m) - \phi_{c_dc_s}(z, m) - \phi_{c_sc_d}(z, m) - \phi_{c_dc_n}(z, m) - \phi_{c_nc_d}(z, m). \quad (30)$$

If $G_o(z, m)$ is the over-all modified pulse-transfer function of the system

$$G_o(s) = W^*(s)G(s) \quad (31)$$

where

$$W^*(s) = D^*(s)/[1 + D^*(s)G^*(s)] \quad (32)$$

we have

$$G_o(z, m) = W(z)G(z, m). \quad (33)$$

The block diagram of Fig. 2 is redrawn in Fig. 3 with the feedback

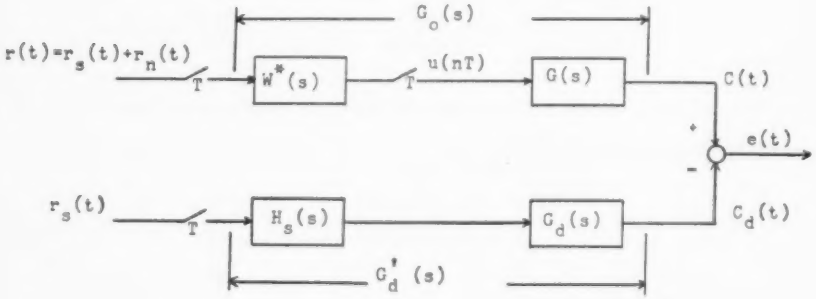


FIG. 3.

system represented by its open-loop equivalent and the desired transfer function $G_d(s)$ preceded by a sampler and an ideal desampler. In order for the signal input to $G_d(s)$ to have the same statistical properties, it

can be shown that the transfer function $H_s(s)$ of the ideal desampler is given by

$$H_s(s) = \phi_{r_s r_s}(s)/\phi^*_{r_s r_s}(s). \tag{34}$$

In view of this, (30) reduces to

$$\begin{aligned}\phi_{ee}(z, m) = {} & [W(z)W(z^{-1})G(z, m)G(z^{-1}, m) \\ & - W(z)G(z, m)G_d'(z^{-1}, m) \\ & - W(z^{-1})G(z^{-1}, m)G_d'(z^{-1}, m) \\ & - G_d'(z, m)G_d'(z^{-1}, m)]\phi_{r_s r_s}(z) \\ & + [W(z)W(z^{-1})G(z, m)G(z^{-1}, m) \\ & - W(z)G(z, m)G_d'(z^{-1}, m)]\phi_{r_s r_n}(z) \\ & + [W(z)W(z^{-1})G(z, m)G(z^{-1}, m) \\ & - W(z^{-1})G(z^{-1}, m)G_d'(z, m)]\phi_{r_n r_s}(z) \\ & + W(z)W(z^{-1})G(z, m)G(z^{-1}, m)\phi_{r_n r_n}(z)\end{aligned} \tag{35}$$

where $G_d'(z, m)$ is the modified z-transform associated with $G_d'(s) = H_s(s)G_d(s)$ and $\phi_{r_s r_s}(z)$, $\phi_{r_n r_n}(z)$, $\phi_{r_s r_n}(z)$ and $\phi_{r_n r_s}(z)$ denote the pulse-spectral densities and the cross pulse-spectral densities for the input signal $r_s(t)$ and the input noise $r_n(t)$.

From Fig. 2,

$$\overline{u^2(nT)} = \frac{1}{2\pi j}\oint_\Gamma \phi_{uu}(z)z^{-1}dz \tag{36}$$

where

$$\phi_{uu}(z) = W(z)W(z^{-1})[\phi_{r_s r_s}(z) + \phi_{r_n r_n}(z) + \phi_{r_s r_n}(z) + \phi_{r_n r_s}(z)]. \tag{37}$$

Now form the functional

$$F = \overline{e^2(t)} + \lambda\,\overline{u^2(nT)} \tag{38}$$

where

$$\overline{e^2(t)} = \int_0^1 \overline{e^2(nT, m)}dm \tag{39}$$

$$= \frac{1}{2\pi j}\oint_\Gamma z^{-1}dz \int_0^1 \phi_{ee}(z, m)dm. \tag{40}$$

We have

$$\begin{aligned}\int_0^1 \phi_{ee}(z, m)dm = {} & W(z)W(z^{-1})K_1(z)\phi(z) \\ & - W(z)K_2(z)\phi_1(z) - W(z^{-1})K_2(z^{-1})\phi_2(z) \\ & + K_3(z)\phi_{r_s r_s}(z)\end{aligned} \tag{41}$$

where

$$K_1(z) = \int_0^1 G(z, m)G(z^{-1}, m)dm \tag{42}$$

$$K_2(z) = \int_0^1 G(z, m)G_d'(z^{-1}, m)dm \tag{43}$$

$$K_3(z) = \int_0^1 G_d'(z, m)G_d'(z^{-1}, m)dm \tag{44}$$

$$\phi(z) = \phi_{r_s r_s}(z) + \phi_{r_s r_n}(z) + \phi_{r_n r_s}(z) + \phi_{r_n r_n}(z) \tag{45}$$

$$\phi_1(z) = \phi_{r_s r_s}(z) + \phi_{r_s r_n}(z) \tag{46}$$

and

$$\phi_2(z) = \phi_{r_s r_s}(z) + \phi_{r_n r_s}(z). \tag{47}$$

The minimization of (38) subject to the constraint that

$$\overline{u^2(nT)} \leq \sigma^2 \tag{48}$$

is solved as before. The result is

$$W(z) = \frac{\left\{ \dfrac{K_2(z^{-1})[\phi_{r_s r_s}(z) + \phi_{r_n r_s}(z)]}{B^-(z)\phi^-(z)} \right\}_+}{B^+(z)\phi^+(z)} \tag{49}$$

where $B(z) = [\lambda + K_1(z)]$.

SYSTEMS WITH NOISE IN THE CONTROL LOOP

With reference to Fig. 4, the system error is given by

$$\begin{aligned} e(t) &= c_d(t) - c(t) \\ &= c_d(t) - c_s(t) - c_n(t). \end{aligned} \tag{50}$$

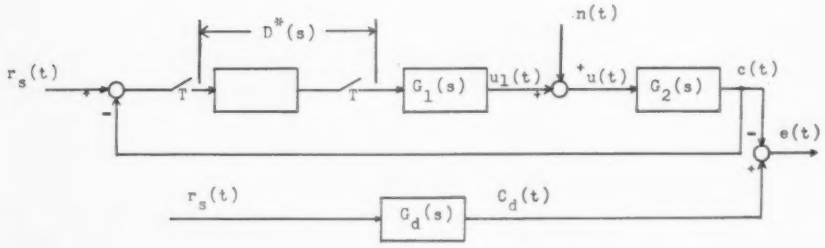


FIG. 4.

In terms of the error samples,

$$e(nT, m) = c_d(nT, m) - c_s(nT, m) - c_n(nT, m). \tag{51}$$

The case where the input signal $r_s(t)$ and the noise $n(t)$ are uncorrelated will be considered. The mean-square value of the error samples is

$$\overline{e^2(nT, m)} = \overline{c_d^2(nT, m)} + \overline{c_s^2(nT, m)} + \overline{c_n^2(nT, m)} - \overline{c_s(nT, m)c_d(nT, m)} - \overline{c_d(nT, m)c_s(nT, m)}. \tag{52}$$

The mean-square value of the error samples is given by (29), where

$$\phi_{ee}(z, m) = \phi_{c_dc_d}(z, m) + \phi_{c_sc_s}(z, m) + \phi_{c_nc_n}(z, m) - \phi_{c_sc_d}(z, m) - \phi_{c_dc_s}(z, m). \tag{53}$$

Figure 4 is redrawn in Fig. 5 to simplify the evaluation of $\phi_{ee}(z, m)$, where $H_s(s)$ and $H_n(s)$ are the transfer functions of the ideal desamplers

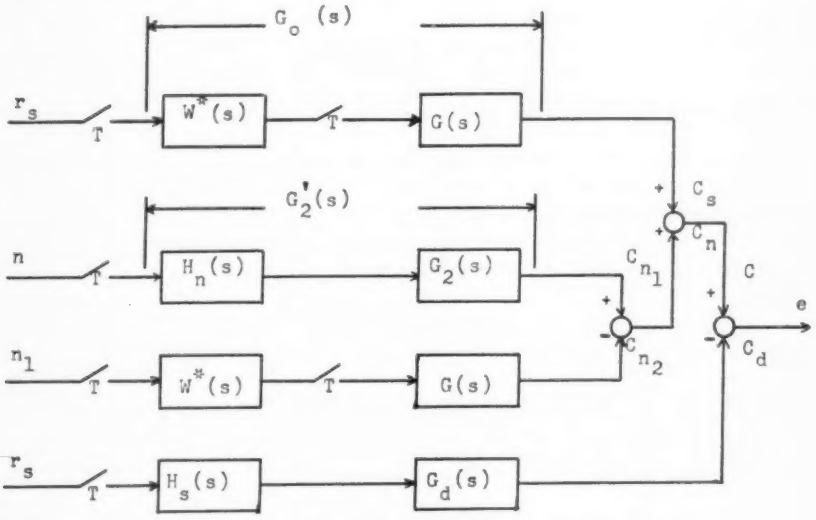


FIG. 5.

for the signal $r_s^*(t)$ and the noise $n^*(t)$. The transfer function $H_s(s)$ is given by (34) and it can be shown that

$$H_n(s) = \frac{\phi_{nn}(s)}{\phi_{nn}^*(s)}. \tag{54}$$

The modified pulse-spectral densities of (53) are given by

$$\phi_{c_s c_s}(z, m) = W(z)W(z^{-1})G(z, m)G(z^{-1}, m)\phi_{r_s r_s}(z) \tag{55}$$

$$\phi_{c_d c_d}(z, m) = G_d'(z, m)G_d'(z^{-1}, m)\phi_{r_s r_s}(z) \tag{56}$$

$$\phi_{c_s c_d}(z, m) = W(z^{-1}, m)G(z^{-1}, m)G_d'(z, m)\phi_{r_s r_s}(z) \tag{57}$$

$$\phi_{c_d c_s}(z, m) = W(z, m)G(z, m)G_d'(z^{-1}, m)\phi_{r_s r_s}(z) \tag{58}$$

$$\begin{aligned}\phi_{c_n c_n}(z, m) = {} & W(z)W(z^{-1})G(z, m)G(z^{-1}, m)\phi_{n_1 n_1}(z) \\ & - W(z)G(z, m)G_2'(z^{-1}, m)\phi_{n n_1}(z) \\ & - W(z^{-1})G(z^{-1}, m)G_2'(z, m)\phi_{n_1 n}(z) \\ & + G_2'(z, m)G_2'(z^{-1}, m)\phi_{n n}(z).\end{aligned} \tag{59}$$

In (59), $G_2'(z, m)$ is the modified z-transform associated with

$$G_2'(s) = H_n(s)G_2(s) \tag{60}$$

and

$$N_1(s) = c_{n_1}(s). \tag{61}$$

In Fig. 4,

$$G(s) = G_1(s)G_2(s). \tag{62}$$

Referring to Fig. 4, the constraint is

$$\overline{u^2(t)} \leq \sigma^2 \tag{63a}$$

which can be written as

$$\overline{u_1^2(t)} \leq \sigma_1^2 \tag{63b}$$

where

$$\sigma_1^2 = \sigma^2 - \overline{n^2(t)} \tag{63c}$$

and

$$\overline{u_1^2(t)} = \frac{1}{2\pi j}\oint_\Gamma \phi_{u_1 u_1}(z, m)z^{-1}dz. \tag{64}$$

In (64), $\phi_{u_1 u_1}(z, m)$ is given by

$$\phi_{u_1 u_1}(z, m) = W(z)W(z^{-1})G_1(z, m)G_1(z^{-1}, m)\phi_{r_s r_s}(z). \tag{65}$$

Application of the calculus of variations to the functional $F = \overline{e^2(t)} + \lambda\,\overline{u_1^2(t)}$ yields,

$$W(z) = \frac{\left\{\dfrac{K_2(z^{-1})\phi_{r_s r_s}(z) + K_4(z^{-1})\phi_{n_1 n}(z)}{B^-(z)}\right\}_+}{B^+(z)} \tag{66}$$

where

$$B(z) = K_1(z)[\phi_{r_d r_d}(z) + \phi_{nn}(z)] + \lambda K_6(z)\phi_{r_d r_d}(z) \tag{67}$$

$$K_4(z) = \int_0^1 G(z, m)G_2'(z^{-1}, m)dm \tag{68}$$

and

$$K_6(z) = \int_0^1 G_1(z, m)G_1(z^{-1}, m)dm. \tag{69}$$

Clearly, the technique could be extended to cases where constraints are placed upon more than one signal in the control loop and the signal and noise are correlated.

CONCLUSION

This paper introduces a simplified systematic procedure for statistical design of discrete-data control systems when the control signal is constrained. The system performance is optimized in the sense of minimum mean-square error. The system inputs are assumed to be functions of stationary random processes. Although only systems with one constrained signal are discussed, the design technique could be easily extended to cases where constraints are placed upon more than one signal in the control loop. The application of this procedure to optimum design for minimum integral-square error and subject to power limitation is apparent, if use is made of the results obtained in reference 3. Furthermore, this design technique may be extended to certain types of nonlinear control systems, provided that the nonlinearity can be described or approximated as constraints on control signals.

Acknowledgment

The work reported here was supported in part by the National Science Foundation and the Office of Naval Research through the Information Systems Branch.

REFERENCES

(1) J. T. TOU, "Digital and Sampled-Data Control Systems," New York, McGraw-Hill Book Co., Inc., 1959.

(2) J. T. TOU, "Statistical Design of Digital Control Systems," *IRE Trans. on Automatic Control*, Vol. AC-5, No. 4, pp. 290–297 (1960).

(3) J. T. TOU, "Statistical Design of Linear Discrete-Data Control Systems via the Modified z-Transform Method," JOUR. FRANKLIN INST., Vol. 271, pp. 249–262 (1961).

(4) G. FRANKLIN, "Linear Filtering of Sampled-Data," *IRE Convention Record*, Pt. 4, pp. 119–128 (1955).

(5) R. M. STEWART, "Statistical Design and Evaluation of Filters for the Restoration of Sampled Data," *IRE Proc.*, Vol. 44, pp. 253–257 (1956).
(6) A. S. ROBINSON, "The Synthesis of Computer-Limited Sampled-Data Control Systems," AIEE Special publication T-101, May 1958, pp. 77–87.
(7) S. S. L. CHANG, "Statistical Design Theory for Digital-Controlled Continuous Systems," *AIEE Trans.*, Vol. 77, pt. II, pp. 191–201 (1958).
(8) H. S. TSIEN, "Engineering Cybernetics," New York, McGraw-Hill Book Co., Inc., 1954.
(9) G. C. NEWTON, L. A. GOULD AND J. F. KAISER, "Analytical Design of Linear Feedback Control," New York, John Wiley & Sons, Inc., 1957.

STRESSES IN CYLINDRICALLY SYMMETRIC MEMBRANES REINFORCED WITH EXTENSIBLE CORDS

BY

W. F. AMES[1]

SUMMARY

The theories of Rivlin and Hofferberth on the equilibrium shape and stresses in deformed membranes reinforced by cords are limited to inextensible cords and to an average cord-extension ratio, respectively. The results of this paper remove these limitations for the case of cylindrically symmetrical deformation. The equilibrium shape and the stress problems are solved for the deformed membrane.

1. INTRODUCTION

In recent years there have been a number of quite successful efforts to apply the membrane and shell theories of elasticity to the analysis of the deformation and stresses in reinforced structures. Of special interest in this area is the work of Adkins (1, 2)[2] Rivlin (3, 4), Adkins and Rivlin (5), Hofferberth (6, 7, 8) and Lauterbach and Ames (9). In their earlier papers (1, 2, 5), Adkins and Rivlin considered the deformation of sheets and shells of a material consisting of incompressible isotropic elastic material reinforced by inextensible cords. The presence of the elastic material made it difficult to solve any but the simplest problems.

A number of technological problems, of interest to engineers, are such that either the elastic material is absent altogether (for example, cloth, canvas, etc.) or the elastic material plays a subordinate role in the determination of the deformation and stress characteristics of the system. Examples of the latter case include the pneumatic tire, balloons, radomes and so forth. Realization of this subordinate position of the elastic material prompted Rivlin (3) to consider the plain strain of a net formed by two families of straight parallel inextensible cords. As an outgrowth of this work a fundamental paper by Rivlin (4) considered the deformation of a membrane formed by inextensible cords. A direct application to engineering structures is possible from this theory, especially that case of cylindrically symmetric deformation which concerns us here.

Paralleling, but independent of, the work of Rivlin, were the efforts of Hofferberth (6, 7, 8). Hofferberth was entirely concerned with the development of an analytic tire theory based upon membrane theory. Consequently, his results are not as general as those of Rivlin. Early

[1] Department of Mechanical Engineering, University of Delaware, Newark, Del.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

in this work Hofferberth's results, on the membrane equilibrium shape, became known to the present author. By use of these results the stress problem in cylindrically symmetric deformation (tirelike structures) was solved independent of the work of Rivlin. Lauterbach and Ames (9) demonstrated that the tire stresses in the two principal directions and the cord tension are quite variable which results from the expansion process.

Also to be mentioned in this area of work is the book of the Russian V. L. Biederman on "Tire Design."

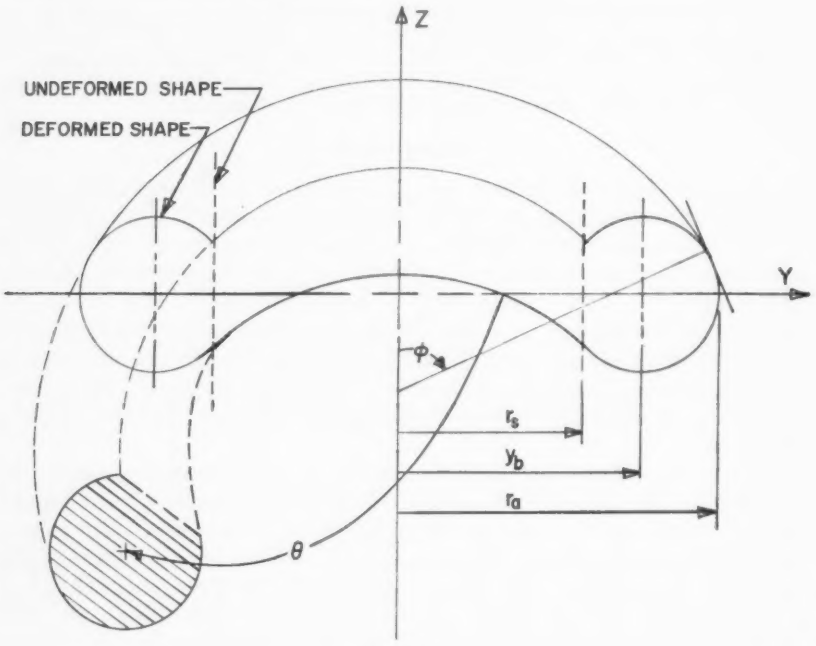


FIG. 1.

The Rivlin theory has the defect that the cords are assumed inextensible. Hofferberth's theory purports to remove this limitation but his membrane shape includes a cord-extension ratio whose dependence on the horizontal dimension y is unknown. Thus his results are limited to the case of using an average cord-extension ratio. The departure from reality, in this case, is not severe since the cord strain realized in tires is of the order of 2–4 per cent resulting from standard inflation pressures. However, in other applications, such as reinforced tubing, this limitation may be extremely important.

The present paper is concerned with removing the inextensible limitation of previous work for the case of cylindrically symmetric

deformation. A Hookean relation is assumed between the cord strain and the cord stress. The cord extensibility is then obtained in dependence upon dimension and the equation for the shape of the deformed membrane is derived. Finally the stress problem is solved illustrating its dependence upon physical parameters, membrane shape and type of loading.

2. BASIC EQUATIONS

The notation used throughout the paper is consistent with that of the previous paper (**9**) which agrees with that of Hofferberth (**8**).

The deformed membrane is a surface of revolution so that an orthogonal net with coordinates (ϕ, θ) is chosen on the surface. Rectangular coordinates (y, z) are chosen in the plane as illustrated in Fig. 1. We shall refer to ϕ as a meridian coordinate and θ as a circumferential coordinate.

From (**4**, **8**, **9**) the meridian equation is

$$\frac{dz}{dy} = \tan \phi \tag{1}$$

and the shape equilibrium equation of the membrane is

$$\cot \phi \frac{d\phi}{dy} + \frac{\cot^2 \beta}{y} = \frac{2y}{y^2 - y_B^2} \tag{2}$$

where $\beta = \beta(y)$ is the angle of the deformed membrane cords with the circumferential circles and α is the constant cord angle with the circumference of the membrane in the predeformed cylindrical shape.

From the cord network considerations of Rivlin (**3**) as modified by Hofferberth (**8**) we have the relation

$$y = \frac{c_t r_a \cos \beta}{\cos \alpha} \tag{3}$$

where c_t is the cord-extension ratio and the other parameters are defined in Fig. 1.

From (**9**) the cord tension τ is expressible in the form

$$\tau = \frac{(y^2 - y_B^2) p c_t \cot \beta}{2N n_0 y \sin \phi \sin \alpha \cos \alpha} \tag{4}$$

where p = internal pressure, n_0 = number of cords per inch of material before deformation and N = number of cord layers.

Finally, from the membrane theory of elasticity, the equations of

force equilibrium for the membrane are

$$\frac{1}{r_1}\frac{\partial(Ty)}{\partial\phi} + \frac{\partial N_\theta}{\partial\theta} + T\cos\phi + p_\theta y = 0$$

$$\frac{1}{r_1}\frac{\partial(N_\phi y)}{\partial\phi} + \frac{\partial T}{\partial\theta} - N_\theta\cos\phi + p_\phi y = 0 \tag{5}$$

$$\frac{N_\theta}{r_2} + \frac{N_\phi}{r_1} = p_\nu$$

where N_θ = stress component in the circumferential direction, N_ϕ = stress component in the meridian direction, $T = N_{\phi\theta} = N_{\theta\phi}$ = shear component, p_θ = load component in the circumferential direction, p_ϕ = load component in the meridian direction, p_ν = load component normal to the surface, and r_1, r_2 = radii of curvature in the meridian and parallel direction, respectively.

3. CORD EXTENSION DEPENDENCY ON y

We now assume a linear relation between the cord stress τ and cord strain ϵ, $\epsilon = A\tau$ so that the cord extension ratio can be written as

$$c_t = 1 + \epsilon = 1 + A\tau \tag{6}$$

where A is the appropriate physical constant. With the assumption of (6) the occurrence of c_t in the shape integral (see **8**, **9**) can now be eliminated by obtaining the dependence of c_t on y.

From (3)

$$c_t = \frac{y\cos\alpha}{r_s\cos\beta} = 1 + A\tau. \tag{7}$$

By using (4) and (7) there results, after some algebra,

$$\frac{1}{A}\left[\frac{y\cos\alpha}{r_s\cos\beta} - 1\right] = \tau = \frac{(y^2 - y_B{}^2)p}{2Nn_0 r_s\sin\beta\sin\phi\sin\alpha}. \tag{8}$$

Solving for $\sin\phi$, we have the result

$$\sin\phi = \gamma\frac{(y^2 - y_B{}^2)r_s\cos\beta}{\sin\beta(y\cos\alpha - r_s\cos\beta)} \tag{9}$$

where

$$\gamma = \frac{p}{2Nn_0 r_s\sin\alpha A} \tag{10}$$

and (9) is independent of c_t but does still depend upon β. The dependence of β upon y must be found in order that $\sin\phi$ and c_t will have an exhibited relationship with y.

Upon differentiation of (9) with respect to y and division by $\sin\phi$ there results

$$\cot\phi\,\frac{d\phi}{dy} = \frac{2y}{y^2 - y_B{}^2} - (\tan\beta + \cot\beta)\,\frac{d\beta}{dy} - \frac{\cos\alpha + r_s \sin\beta\,\dfrac{d\beta}{dy}}{y\cos\alpha - r_s\cos\beta}. \quad (11)$$

But, substituting (2) in (11) there results

$$\frac{\cot^2\beta}{y} = (\tan\beta + \cot\beta)\,\frac{d\beta}{dy} + \frac{\cos\alpha + r_s\sin\beta\,\dfrac{d\beta}{dy}}{y\cos\alpha - r_s\cos\beta} \quad (12)$$

which is the differential equation for β as a function of y. The solution of (12) is facilitated by the introduction of the new variables

$$\begin{aligned} u &= \cos\beta \\ v &= \frac{y\cos\alpha}{r_s} \end{aligned} \quad (13)$$

whereupon (12) becomes

$$\frac{du}{dv} = \frac{u[v(2u^2 - 1) - u^3]}{v(u^3 - v)} \quad (14)$$

which has the solution

$$uv - \frac{v^2(2u^2 - 1)}{2u^2} = C. \quad (15)$$

When $y = r_s$, no expansion of the cord matrix has taken place from the undeformed cylindrical state. Therefore, at $y = r_s$, $\beta = \alpha$ and $u_s = v_s = \cos\alpha$. It is easily seen that $C = \frac{1}{2}$ and (15) becomes

$$2u^2v - (2v^2 + 1)u^2 + v^2 = 0 \quad (16)$$

which is the implicit expression for $u = \cos\beta$ as a function of v, and hence of y. Clearly $0 \leq u \leq 1$ so that the desired root of (16) lies between 0 and 1. The function given by the left-hand side of (16) is easily shown to have a minimum (of interest here) at $u = \dfrac{2v^2 + 1}{3v}$ with the minimum value

$$P_m = v^2 - \frac{(2v^2 + 1)^3}{27v^2}.$$

Returning to (9) we write

$$\sin\phi = \gamma\,\frac{(y^2 - y_B{}^2)u}{(1 - u^2)^{1/2}(v - u)} = B\,\frac{(v^2 - v_B{}^2)u}{(1 - u^2)^{1/2}(v - u)} \quad (17)$$

where $B = \dfrac{pr_s}{2ANn_0\sin\alpha\cos^2\alpha}$ and $v_B = \dfrac{y_B\cos\alpha}{r_s}$. As v varies from

$\cos\alpha$ to $r_a\cos\alpha/r_s$, the corresponding value of $u = \cos\beta$ may be found by numerical solution of the cubic equation (16). Hence c_t may be found from (7).

When (17) is substituted into (1) the shape of the deformed membrane becomes

$$z = \mp \int_{v_a}^{v} \frac{r_s B(v^2 - v_B{}^2)u\,dv}{\cos\alpha[(1-u^2)(v-u)^2 - B^2(v^2-v_B{}^2)^2u^2]^{1/2}} \tag{18}$$

where the negative sign applies when z is positive and the positive sign when z is negative. The quantity $v_a = \dfrac{r_a\cos\alpha}{r_s}$ is introduced as the lower limit since $z(r_a) = 0$. This integral is hyperelliptic and evaluation numerically is necessary.

4. MEMBRANE STRESSES

The solution of the membrane equilibrium equations (5) has already been given in general form in (9). If we denote by $f(y)$ the function under the integral sign in (18), that is $f(y) = \tan\phi$, we have from (9) that the stress N_θ in the circumferential direction is

$$N_\theta = \frac{py\sqrt{1+f^2(y)}}{f(y)}\left[1 - \frac{1}{2y}\frac{(y^2 - y_B{}^2)\dfrac{df}{dy}}{f(y)(1+f^2(y))}\right] \tag{19}$$

and the stress N_ϕ in the meridian direction is given by

$$N_\phi = \frac{p}{2y}\frac{(y^2 - y_B{}^2)\sqrt{1+f^2(y)}}{f(y)}. \tag{20}$$

Finally, the tension τ in the cords as a function of y is given by

$$\tau = \frac{yN_\phi}{Nn_0r_s\sin\alpha\sin\beta} = \frac{p(y^2 - y_B{}^2)\sqrt{1+f^2(y)}}{2Nn_0r_s\sin\alpha f(y)\sqrt{1-\cos^2\beta}}$$

where $\cos\beta$ is calculated from (16).

REFERENCES

(1) J. E. ADKINS, *J. Rat. Mech. Anal.*, Vol. 5, p. 189 (1956).
(2) J. E. ADKINS, *Phil. Trans. Roy. Soc. A*, Vol. 249, p. 125 (1956).
(3) R. S. RIVLIN, *J. Rat. Mech. Anal.*, Vol. 4, p. 951 (1955).
(4) R. S. RIVLIN, *Arch. Rat. Mech. Anal.*, Vol. 2, p. 447 (1959).
(5) J. E. ADKINS AND R. S. RIVLIN, *Phil. Trans. Roy. Soc., A*, Vol. 248, p. 201 (1955).
(6) W. HOFFERBERTH, *Kautschuk u. Gummi*, Vol. 8, Sonderheft "Der Reifen" 3 (1955).
(7) W. HOFFERBERTH, *Kautschuk u. Gummi*, Vol. 8, p. 124 (1955).
(8) W. HOFFERBERTH, *Kautschuk u. Gummi*, Vol. 9, p. 225 (1956).
(9) H. G. LAUTERBACH AND W. F. AMES, *Tex. Res. J.*, Vol. 29, p. 890 (1959).

ANALYTICAL SOLUTIONS FOR ROTARY MATRIX WIRE SCREEN HEAT EXCHANGERS

BY

W. H. T. LOH[1]

ABSTRACT

Instead of the partial differential equation commonly used for rotary matrix heat exchangers, an ordinary differential equation is developed for a single stage rotary wire screen heat exchanger. This equation results in an exact solution for temperature effectiveness. The solution for a multi-screen system is then developed by an energy balance of each individual single screen. Based on analytical solutions obtained here, design charts for rotary multilayer wire screen heat exchangers are presented.

INTRODUCTION

The rotary matrix heat exchanger, developed in early 1922, embodies a moving heat conveyor, heated in the hot stream and cooled in the cold stream. Heat is thus transferred by rotating the conveyor through the two streams. The heat conveyor originally proposed was a rotary matrix made of plain metal plate surfaces. Improvements in heat transfer were effected through the use of corrugated plate-fin type surfaces. Further improvements may be obtained by using multilayer individual wire screen as the conveyor.

Analytical studies relating to rotating heat exchangers have been made by Nussett, Hausen, Boestad, Saunders and many others.

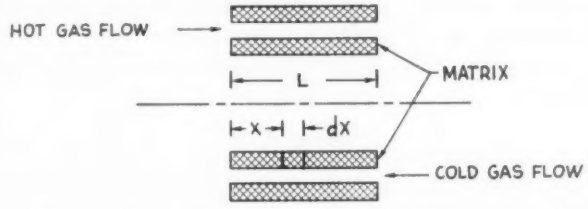


FIG. 1. Flow passage length for generalized rotary matrix.

However, these authors have concentrated on the solution of the partial differential equation which was derived primarily for the generalized rotary matrix along the lines of the original proposed arrangements. This type of arrangement has a flow passage length (L) as shown in Fig. 1.

It can be seen that the temperature (T) is a function of both time (t)

[1] Chance Vought Research Center, Dallas, Texas.

and space (x), and that the only equation describing this type of matrix is partial differential equation

$$WC_p \frac{\partial T}{\partial x} + \rho A_f C_p \frac{\partial T}{\partial t} = -\frac{M_r C_{pr}}{L} \frac{\partial T_r}{\partial t} = -\frac{hA}{L}(T - T_r). \quad (1)$$

Because of mathematical complexities, solutions were generally obtained either by graphical or numerical methods, since no analytical closed form solution has been found. The present analysis takes a different approach. This approach, primarily for multilayer rotary matrix, starts with a single screen (see Fig. 2).

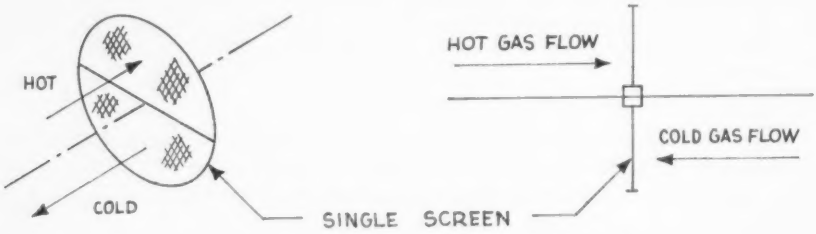


FIG. 2. Flow passage length for single screen = $L \cong 0$.

From Fig. 2, it can be seen that the temperature (T) is a function of time (t) only. Therefore, an ordinary differential equation was developed, resulting in an exact and closed form solution. The multilayer screen system was then developed by an energy balance of each individual single screen. Since this approach led to a closed form analytical solution for the rotating multilayer wire screen heat exchanger, it was then possible to develop design charts for this type of heat exchangers.

ASSUMPTIONS

1. Gas and material properties such as ρ, C_p, k, μ, P_r, etc. are treated as constants.
2. Heat transfer through conduction and radiation is small compared with heat transfer through convection; conduction and radiation heat transfer is, therefore, neglected.
3. Variations in the gas temperature ahead of each individual screen are small when compared with the gas temperature itself, hence the gas temperature ahead of each screen is treated as uniform, as is the gas temperature after each screen.
4. The size of wire is so small that it may be treated as a point for gas flow across it.

ANALYSIS

Equal Mass Flow for Hot and Cold Streams

Equal mass flow in equal cross-sectional areas results in equal values of G, and consequently, equal values of h in both hot and cold streams.

1. Wire Temperature vs. Time

For a cross-wire screen element as shown in Fig. 3, the energy equation gives

$$\frac{dT_w}{dt} + \left(\frac{4h}{\rho_w C_{p_w} d}\right) T_w = \left(\frac{4h}{\rho_w C_{p_w} d}\right) T. \tag{2}$$

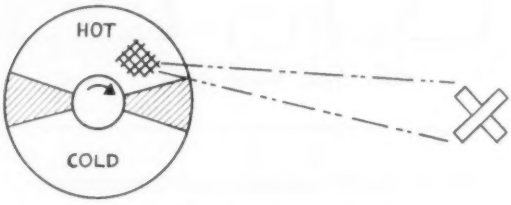


FIG. 3. Cross-wire screen element.

As the wire element rotates from hot stream to cold stream, the surrounding gas temperature (T), shown in Fig. 4 can be expressed by the Fourier series

$$T = \sum_{n=\text{odd}}^{\infty} \frac{2}{\pi}\left(\frac{T_h - T_c}{n}\right) \sin n\omega t + \left(\frac{T_h + T_c}{2}\right). \tag{3}$$

Solving Eqs. 2 and 3 gives

$$T_w = \left\{T_{w_i} - b_0 + \sum_{n=\text{odd}}^{\infty} a_n \left[\frac{(n\omega/c)}{1 + (n\omega/c)^2}\right]\right\} e^{-ct} + \left\{b_0 + \sum_{n=\text{odd}}^{\infty} a_n \left[\frac{1}{\sqrt{1 + (n\omega/c)^2}}\right] \sin (n\omega t - \phi_n)\right\}. \tag{4}$$

Equation 4 describes the complete time history of the wire temperature from the very start to steady state cyclic operations. Since the transient portion becomes negligibly small after an elapsed time approximately equal to $(3/C)$ seconds from the very start of the operation, only the steady state portion of the complete solution is really needed. This steady state solution in non-dimensional form is

$$\frac{T_w - b_0}{T_h - T_c} = \sum_{n=\text{odd}}^{\infty} \left[\left(\frac{2}{\pi}\right) \frac{1}{n\sqrt{1 + (n\omega/c)^2}}\right] \sin (n\omega t - \phi_n). \tag{5}$$

Equation 5 describes the complete wire element steady state cyclic temperature variations in terms of wire geometry, wire material properties, hot and cold stream temperatures, heat transfer coefficients, and revolutions per second of the wire screen rotor.

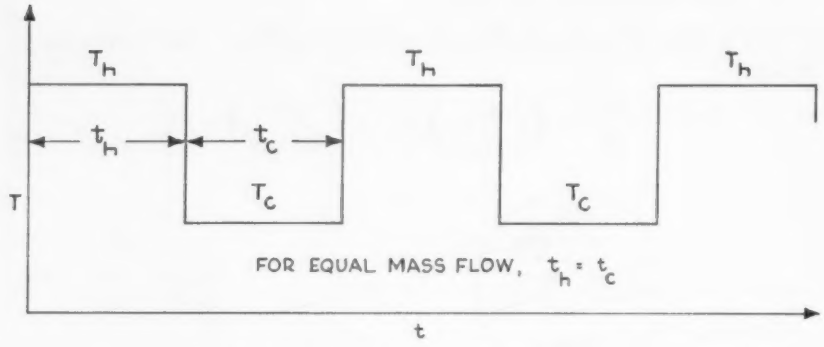


FIG. 4. Gas temperature in rotating wire element.

2. Cyclic Wire Temperature Rise

After steady state heat transfer is attained, the zero time ($t = 0$) can be set at the instant that the wire element enters the hot stream. The wire element leaves the hot stream at $t = t_h$. If T_{w_0} indicates the wire element temperature in °F. at $t = 0$ and T_{w_τ} indicates the wire element temperature in °F. at $t = t_h$, then from Eq. 5 the values of T_{w_0} and T_{w_τ} can be calculated as:

$$\frac{T_{w_0} - b_0}{T_h - T_c} = \sum_{n=\text{odd}}^{\infty} \mu_n \sin(-\phi_n) = -\sum_{n=\text{odd}}^{\infty} \mu_n \sin \phi_n \tag{6}$$

$$\frac{T_{w_\tau} - b_0}{T_h - T_c} = \sum_{n=\text{odd}}^{\infty} \mu_n \sin(n\pi - \phi_n) = \sum_{n=\text{odd}}^{\infty} \mu_n \sin \phi_n \tag{7}$$

$$\mu_n = \left(\frac{2}{\pi}\right) \frac{1}{n\sqrt{1 + (n\omega/c)^2}} \qquad \phi_n = \tan^{-1}\left(\frac{n\omega}{c}\right).$$

Consequently, the wire temperature rise ΔT_w from T_{w_0} to T_{w_τ} during the passage of the wire element through the hot stream section is

$$\frac{\Delta T_w}{T_h - T_c} = \frac{T_{w_\tau} - T_{w_0}}{T_h - T_c} = \sum_{n=\text{odd}}^{\infty} 2\mu_n \sin \phi_n. \tag{8}$$

3. Temperature Effectiveness of a Single Rotating Wire Screen

The temperature effectiveness η_s of a single rotating wire screen is defined as

$$\eta_s = \frac{T_h - T_{h_e}}{T_h - T_c}. \tag{9}$$

Using the energy equation, one obtains

$$W_w C_{p_w}(T_{w_s} - T_{w_0}) = W_h C_{p_h}(T_h - T_{h_e}). \tag{10}$$

From geometry and the equation of continuity, one obtains

$$W_w = \left\{ \frac{\frac{\pi}{4}(D_0)^2\left[1 - \left(\frac{D_i}{D_0}\right)^2\right]}{s^2} \right\}\left[2\left(\frac{\pi}{4}d^2\right)s\right]\rho_w N \tag{11}$$

$$W_h = \frac{1}{m}\left\{\frac{\pi}{4}(D_0)^2\left[1 - \left(\frac{D_i}{D_0}\right)^2\right]\left[1 - \frac{m\alpha}{2\pi}\right]\right\}\rho V_a \tag{12}$$

$$G = G_{max}\left(1 - \frac{d}{s}\right)^2. \tag{13}$$

Substituting Eqs. 8, 10, 11, 12, and 13 into Eq. 9 and simplifying, one obtains

$$\eta_s = \frac{1}{\beta}\sum_{n=\text{odd}}^{\infty}\frac{1/\zeta\beta}{1 + (n/\zeta\beta)^2}. \tag{14}$$

Here

$$\zeta = \frac{8}{\pi}\left(\frac{h}{G_{max}C_p}\right)\left[\frac{d/s}{(1 - d/s)^2}\right] \tag{15}$$

$$\beta = \frac{1}{2}\left(\frac{\rho C_p}{\rho_w C_{p_w}}\right)\left(\frac{s}{d}\right)\left(\frac{V_a}{Nd}\right)\left(\frac{1 - m\alpha/2\pi}{m}\right). \tag{16}$$

ζ may be called the "heat transfer geometry parameter," and β may be called the "gas flow geometry parameter." These two nondimensional parameters alone determine the temperature effectiveness of the single layer wire screen. It should be noted that, when the gas flow geometry parameter, β, is small or approaches zero, the temperature effectiveness approaches 1.184ζ.

4. Temperature Effectiveness of a Multilayer Rotating Wire Screen Heat Exchanger

For a single layer wire screen rotor, the results of the expression of temperature effectiveness are the same for both parallel flow and counter

flow heat exchangers. However, the results for parallel flow and counter flow are different for multilayer rotating wire screens. These results are derived separately as follows:

a. Parallel Flow

Figure 5 illustrates a typical arrangement of a multilayer rotating wire screen heat exchanger having n individual screens.

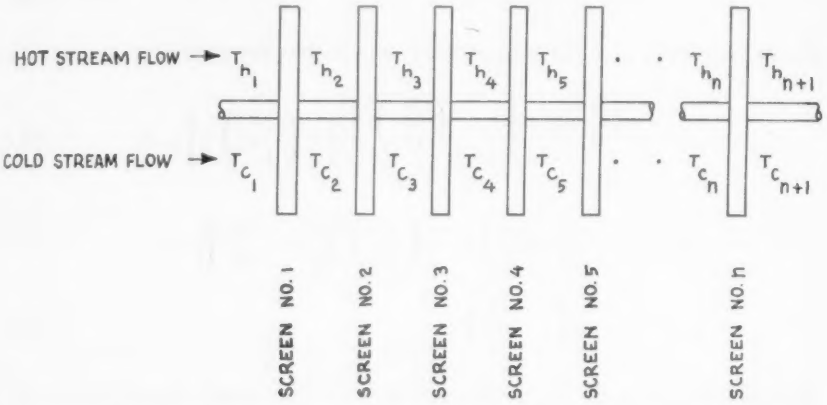


FIG. 5. Typical multilayer rotating wire screen heat exchanger having n individual screens, parallel flow.

By definition,

$$\begin{aligned}
\eta_{s_1} &= \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} & \eta_1 &= \frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}} \\
\eta_{s_2} &= \frac{T_{h_2} - T_{h_3}}{T_{h_2} - T_{c_2}} & \eta_2 &= \frac{T_{h_1} - T_{h_3}}{T_{h_1} - T_{c_1}} \\
\eta_{s_3} &= \frac{T_{h_3} - T_{h_4}}{T_{h_3} - T_{c_3}} & \eta_3 &= \frac{T_{h_1} - T_{h_4}}{T_{h_1} - T_{c_1}} \\
\eta_{s_n} &= \frac{T_{h_n} - T_{h_{n+1}}}{T_{h_n} - T_{c_n}} & \eta_n &= \frac{T_{h_1} - T_{h_{n+1}}}{T_{h_1} - T_{c_1}}
\end{aligned} \tag{17}$$

therefore,

$$\eta_1 = \eta_{s_1} \tag{18}$$

$$\eta_2 = \frac{T_{h_1} - T_{h_3}}{T_{h_1} - T_{c_1}} = \left(\frac{T_{h_1} - T_{h_2}}{T_{h_1} - T_{c_1}}\right) + \left(\frac{T_{h_2} - T_{h_3}}{T_{h_2} - T_{c_2}}\right)\left(\frac{T_{h_2} - T_{c_2}}{T_{h_1} - T_{c_1}}\right), \tag{19}$$

but,

$$T_{h_2} = T_{h_1} - \eta_1(T_{h_1} - T_{c_1}) \tag{20}$$

$$T_{c_2} = T_{c_1} + \eta_1(T_{h_1} - T_{c_1}). \tag{21}$$

Substituting Eqs. 20 and 21 into Eq. 19, one obtains

$$\eta_2 = \eta_1 + \eta_{s_2}(1 - 2\eta_1). \tag{22}$$

Similarly,

$$\eta_3 = \eta_2 + \eta_{s_3}(1 - 2\eta_2) \tag{23}$$

$$\eta_n = \eta_{n-1} + \eta_{s_n}(1 - 2\eta_{n-1}). \tag{24}$$

For the special case of $\eta_{s_1} = \eta_{s_2} = \eta_{s_3} = \eta_{s_n} = \eta_s$, Eqs. 18, 22, 23, and 24 reduce to

$$\left.\begin{aligned} \eta_1 &= \eta_s \\ \eta_2 &= \eta_1 + \eta_s(1 - 2\eta_1) \\ \eta_3 &= \eta_2 + \eta_s(1 - 2\eta_2) \\ &\vdots \\ \eta_n &= \eta_{n-1} + \eta_s(1 - 2\eta_{n-1}) \end{aligned}\right\}. \tag{25}$$

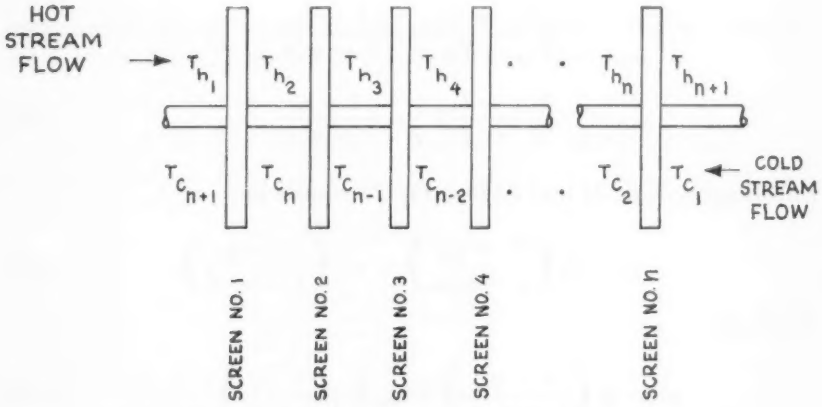


FIG. 6. Typical multilayer rotating wire screen heat exchanger having n individual screens, counter flow.

b. Counter Flow

Figure 6 illustrates a typical arrangement of a multilayer rotating wire screen heat exchanger having n screens. By definition,

$$\eta_{s1} = \frac{T_{h1} - T_{h2}}{T_{h1} - T_{cn}} \qquad \eta_1 = \frac{T_{h1} - T_{h2}}{T_{h1} - T_{cn}}$$

$$\eta_{s2} = \frac{T_{h2} - T_{h3}}{T_{h2} - T_{c,n-1}} \qquad \eta_2 = \frac{T_{h1} - T_{h3}}{T_{h1} - T_{c,n-1}}$$

$$\eta_{s3} = \frac{T_{h3} - T_{h4}}{T_{h3} - T_{c,n-2}} \qquad \eta_3 = \frac{T_{h1} - T_{h4}}{T_{h1} - T_{c,n-2}} \tag{26}$$

$$\eta_{sn} = \frac{T_{hn} - T_{h,n+1}}{T_{hn} - T_{c1}} \qquad \eta_n = \frac{T_{h1} - T_{h,n+1}}{T_{h1} - T_{c1}}$$

Therefore,

$$\eta_1 = \eta_{s1} \tag{27}$$

$$\eta_2 = \eta_1 \left(\frac{T_{h1} - T_{cn}}{T_{h1} - T_{c,n-1}} \right) + \eta_{s2} \left(\frac{T_{h2} - T_{c,n-1}}{T_{h1} - T_{c,n-1}} \right) \tag{28}$$

but,

$$T_{h2} = T_{h1} - \eta_1 (T_{h1} - T_{cn}) \tag{29}$$

$$T_{cn} = T_{c,n-1} + \eta_{s2} (T_{h2} - T_{c,n-1}). \tag{30}$$

Using Eqs. 29 and 30, it is easy to prove that

$$\frac{T_{h1} - T_{cn}}{T_{h1} - T_{c,n-1}} = \frac{1 - \eta_{s2}}{1 - \eta_{s2}\eta_1} \tag{31}$$

$$\frac{T_{h2} - T_{c,n-1}}{T_{h1} - T_{c,n-1}} = \frac{1 - \eta_1}{1 - \eta_{s2}\eta_1}. \tag{32}$$

Substituting Eqs. 31 and 32 into Eq. 28 results in

$$\eta_2 = \eta_1 \left(\frac{1 - \eta_{s2}}{1 - \eta_{s2}\eta_1} \right) + \eta_{s2} \left(\frac{1 - \eta_1}{1 - \eta_{s2}\eta_1} \right). \tag{33}$$

Similarly,

$$\eta_3 = \eta_2 \left(\frac{1 - \eta_{s3}}{1 - \eta_{s3}\eta_2} \right) + \eta_{s3} \left(\frac{1 - \eta_2}{1 - \eta_{s3}\eta_2} \right) \tag{34}$$

$$\eta_n = \eta_{n-1} \left(\frac{1 - \eta_{sn}}{1 - \eta_{sn}\eta_{n-1}} \right) + \eta_{sn} \left(\frac{1 - \eta_{n-1}}{1 - \eta_{sn}\eta_{n-1}} \right). \tag{35}$$

For the special case of $\eta_{s1} = \eta_{s2} = \eta_{s3} = \eta_{sn} = \eta_s$, Eqs. 27, 33, 34 and 35 reduce to

$$\left.\begin{aligned}
\eta_1 &= \eta_s \\
\eta_2 &= \eta_1\left(\frac{1-\eta_s}{1-\eta_s\eta_1}\right) + \eta_s\left(\frac{1-\eta_1}{1-\eta_s\eta_1}\right) \\
\eta_3 &= \eta_2\left(\frac{1-\eta_s}{1-\eta_s\eta_2}\right) + \eta_s\left(\frac{1-\eta_2}{1-\eta_s\eta_2}\right) \\
\eta_4 &= \eta_3\left(\frac{1-\eta_s}{1-\eta_s\eta_3}\right) + \eta_s\left(\frac{1-\eta_3}{1-\eta_s\eta_3}\right) \\
&\vdots \\
\eta_n &= \eta_{n-1}\left(\frac{1-\eta_s}{1-\eta_s\eta_{n-1}}\right) + \eta_s\left(\frac{1-\eta_{n-1}}{1-\eta_s\eta_{n-1}}\right)
\end{aligned}\right\}. \tag{36}$$

UNEQUAL MASS FLOW FOR HOT AND COLD STREAMS

For unequal mass flows, the same value of G in both hot and cold streams may be obtained by designing the ratio of the cross-sectional area of one stream to that of the other so that it is equal to the corresponding mass flow ratio of the two streams. This results in the same values of h in both hot and cold streams.

1. Wire Temperature vs. Time

Equation 2 is also valid for unequal mass flows. The surrounding gas temperature (T), shown in Fig. 7, however, becomes

$$T = b_0' + \sum_{n=1}^{\infty} a_n' \sin n\omega t + \sum_{n=1}^{\infty} b_n' \cos n\omega t$$

$$\left.\begin{aligned}
a_n' &= \frac{T_h - T_c}{n\pi}(1 - \cos nb\pi) \\
b_n' &= \frac{T_h - T_c}{n\pi}(\sin nb\pi) \\
b_0' &= \frac{b}{2}(T_h - T_c) + T_c
\end{aligned}\right\}. \tag{37}$$

The solution of Eqs. 2 and 37 is

$$T_w = A_a' e^{-ct} + b_0' + \sum_{n=1}^{\infty} c_n\left[\frac{1}{\sqrt{1 + (n\omega/c)^2}}\right]\sin(n\omega t + \psi_n - \phi_n). \tag{38}$$

Here

$$c_n = \sqrt{2}\left(\frac{T_h - T_c}{n\pi}\right)\sqrt{1 - \cos nb\pi}$$

$$\phi_n = \tan^{-1}\left(\frac{n\omega}{c}\right) \qquad \psi_n = \tan^{-1}\left[\frac{\sin nb\pi}{1 - \cos nb\pi}\right]. \tag{39}$$

The value of the arbitrary constant A_a' can be determined by setting $t = 0$ and $T_w = T_{w_i}$ in Eq. 38:

$$A_a' = T_{w_i} - b_0' + \sum_{n=1}^{\infty} c_n\left[\frac{1}{\sqrt{1 + (n\omega/c)^2}}\right]\sin(\psi_n - \phi_n). \tag{40}$$

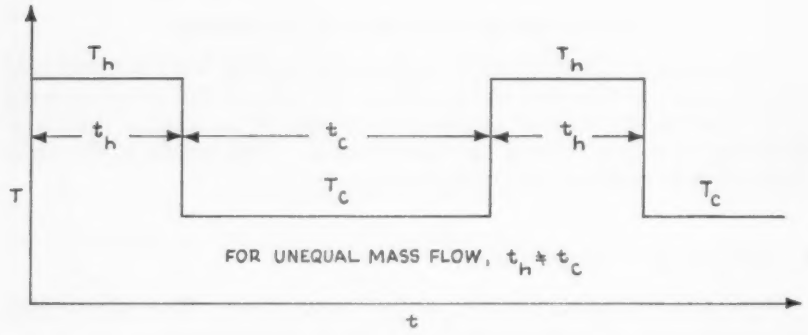


FIG. 7.

After an elapsed time equal to $(3/C)$ seconds from the very start of the operation, the transient state portion becomes negligibly small. Therefore, for practical purposes, only the steady state portion is needed. This solution in nondimensional form is

$$\frac{T_w - b_0'}{T_h - T_c} = \sum_{n=1}^{\infty}\left\{\frac{2}{\pi}\left[\frac{1}{n\sqrt{1 + (n\omega/c)^2}}\right]\left[\sqrt{\frac{1}{2} - \frac{\cos nb\pi}{2}}\right]\right\}\sin(n\omega t + \psi_n - \phi_n). \tag{41}$$

Equation (41) reduces to Eq. (5) when $b = 1$, for equal mass flows.

2. Cyclic Wire Temperature Rise

By employing a procedure similar to the one described for the corresponding portion of equal mass flows, one obtains the wire temperature rise:

$$\frac{\Delta T_w}{T_h - T_c} = \frac{T_{wb\pi} - T_{w0}}{T_h - T_c} = \sum_{n=1}^{\infty} \left[\frac{1}{2} - \frac{\cos nb\pi}{2} \right] 2\mu_n \sin \phi_n. \quad (42)$$

For equal mass flows, Eq. 42 reduces to Eq. 8 when $b = 1$.

3. Temperature Effectiveness of a Single Layer Rotating Wire Screen

By employing a procedure similar to the one described for the corresponding portion of equal mass flows, one obtains

$$\eta_s = \frac{1}{b}\left(\frac{1}{\beta}\right) \sum_{n=1}^{\infty} \left[\frac{1}{2} - \frac{\cos nb\pi}{2} \right] \left[\frac{1/\zeta\beta}{1 + (n/\zeta\beta)^2} \right]. \quad (43)$$

For equal mass flows, Eq. 43 reduces to Eq. 14 when $b = 1$. When $\beta \cong 0$

$$\eta_s \cong \frac{\zeta}{b} \sum_{n=1}^{\infty} \left[\frac{1}{2} - \frac{\cos nb\pi}{2} \right] \left[\frac{1}{n^2} \right]. \quad (44)$$

4. Temperature Effectiveness of a Multilayer Rotating Wire Screen Heat Exchanger

a. Parallel Flows

A procedure similar to that employed for equal mass flows yields

$$\left.\begin{aligned}
\eta_1 &= \eta_{s1} \\
\eta_2 &= \eta_1 + \eta_{s2}\left[1 - 2\eta_1\left(\frac{1}{2} + \frac{f}{2}\right)\right] \\
\eta_3 &= \eta_2 + \eta_{s3}\left[1 - 2\eta_2\left(\frac{1}{2} + \frac{f}{2}\right)\right] \\
\eta_4 &= \eta_3 + \eta_{s4}\left[1 - 2\eta_3\left(\frac{1}{2} + \frac{f}{2}\right)\right] \\
&\vdots \\
\eta_n &= \eta_{n-1} + \eta_{sn}\left[1 - 2\eta_{n-1}\left(\frac{1}{2} + \frac{f}{2}\right)\right]
\end{aligned}\right\}. \quad (45)$$

Where

$$f = \frac{W_h}{W_c} = \frac{b}{2 - b}.$$

For the special case of $\eta_{s1} = \eta_{s2} = \eta_{s3} = \eta_{sn} = \eta_s$, Eq. 45 reduces to

$$\left.\begin{aligned}
\eta_1 &= \eta_s \\
\eta_2 &= \eta_1 + \eta_s\left[1 - 2\eta_1\left(\frac{1}{2} + \frac{f}{2}\right)\right] \\
\eta_3 &= \eta_2 + \eta_s\left[1 - 2\eta_2\left(\frac{1}{2} + \frac{f}{2}\right)\right] \\
\eta_4 &= \eta_3 + \eta_s\left[1 - 2\eta_3\left(\frac{1}{2} + \frac{f}{2}\right)\right] \\
&\vdots \\
\eta_n &= \eta_{n-1} + \eta_s\left[1 - 2\eta_{n-1}\left(\frac{1}{2} + \frac{f}{2}\right)\right]
\end{aligned}\right\}. \tag{46}$$

b. Counter Flows

A procedure similar to that employed for equal mass flows yields

$$\begin{aligned}
\eta_1 &= \eta_{s_1} \\
\eta_2 &= \eta_1\left(\frac{1 - \eta_{s_2} f}{1 - \eta_{s_2}\eta_1 f}\right) + \eta_{s_2}\left(\frac{1 - \eta_1}{1 - \eta_{s_2}\eta_1 f}\right) \\
\eta_3 &= \eta_2\left(\frac{1 - \eta_{s_3} f}{1 - \eta_{s_3}\eta_2 f}\right) + \eta_{s_3}\left(\frac{1 - \eta_2}{1 - \eta_{s_3}\eta_2 f}\right) \\
\eta_n &= \eta_{n-1}\left(\frac{1 - \eta_{s_n} f}{1 - \eta_{s_n}\eta_{n-1} f}\right) + \eta_{s_n}\left(\frac{1 - \eta_{n-1}}{1 - \eta_{s_n}\eta_{n-1} f}\right).
\end{aligned} \tag{47}$$

For the special case of $\eta_{s_1} = \eta_{s_2} = \eta_{s_3} = \eta_s$, Eq. 47 reduces to

$$\left.\begin{aligned}
\eta_1 &= \eta_s \\
\eta_2 &= \eta_1\left(\frac{1 - \eta_s f}{1 - \eta_s\eta_1 f}\right) + \eta_s\left(\frac{1 - \eta_1}{1 - \eta_s\eta_1 f}\right) \\
\eta_3 &= \eta_2\left(\frac{1 - \eta_s f}{1 - \eta_s\eta_2 f}\right) + \eta_s\left(\frac{1 - \eta_2}{1 - \eta_s\eta_2 f}\right) \\
\eta_4 &= \eta_3\left(\frac{1 - \eta_s f}{1 - \eta_s\eta_3 f}\right) + \eta_s\left(\frac{1 - \eta_3}{1 - \eta_s\eta_3 f}\right) \\
&\vdots \\
\eta_n &= \eta_{n-1}\left(\frac{1 - \eta_s f}{1 - \eta_s\eta_{n-1} f}\right) + \eta_s\left(\frac{1 - \eta_{n-1}}{1 - \eta_s\eta_{n-1} f}\right)
\end{aligned}\right\}. \tag{48}$$

DESIGN PROCEDURES AND SAMPLE CALCULATIONS

Equations developed in the Analysis Section may be readily plotted as design charts to establish design criteria. This was done (see Figs. 8–11) for equal mass flows. For unequal mass flows, similar plots may be readily prepared, using the equations for unequal mass flows. These plots are not shown here, but are left to the individual designer to prepare to suit his own applications. For purpose of illustration, the general design procedure is given below:

Given: T_h, T_c, P_h, P_c, W_h, W_c, C_{p_h}, C_{p_c}.
Required: To design a heat exchanger to attain an effectiveness η.
Procedures:

1. Arbitrarily select the following quantities: m, α, n_s, d, s, counter flow or parallel flow, and wire material which determine $\rho_w C_{p_w}$.
2. Read η_s in Fig. 8 or 9 from known values of η and n_s.
3. Select ζ and β in Fig. 10 from known value of η_s. (It is always desirable to have β as close to maximum as possible and ζ as close to minimum as possible.)
4. Calculate $\left(\frac{h}{G_{max}C_p}\right)$ from Eq. 15 and read $\left(\frac{dG_{max}}{\mu}\right)$ in Fig. 11 from known value of $\left(\frac{h}{G_{max}C_p}\right)$.

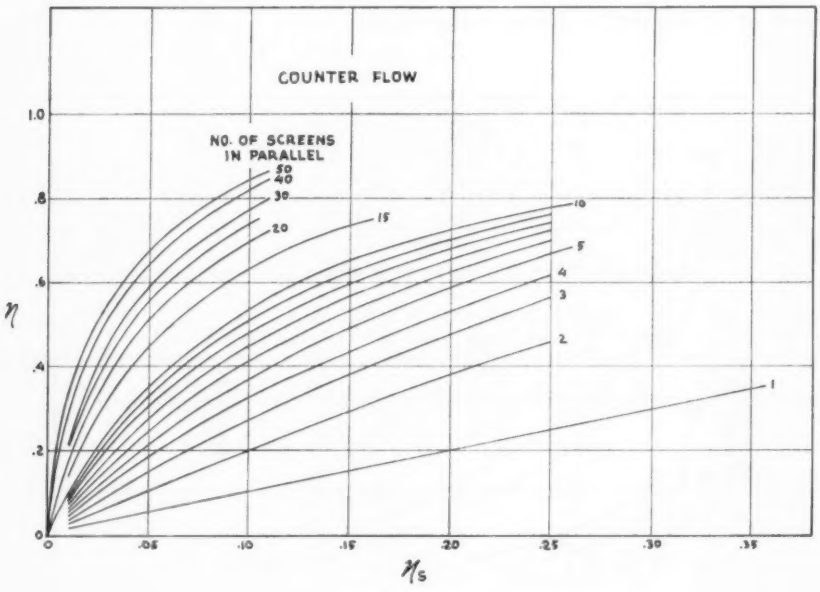


FIG. 8.

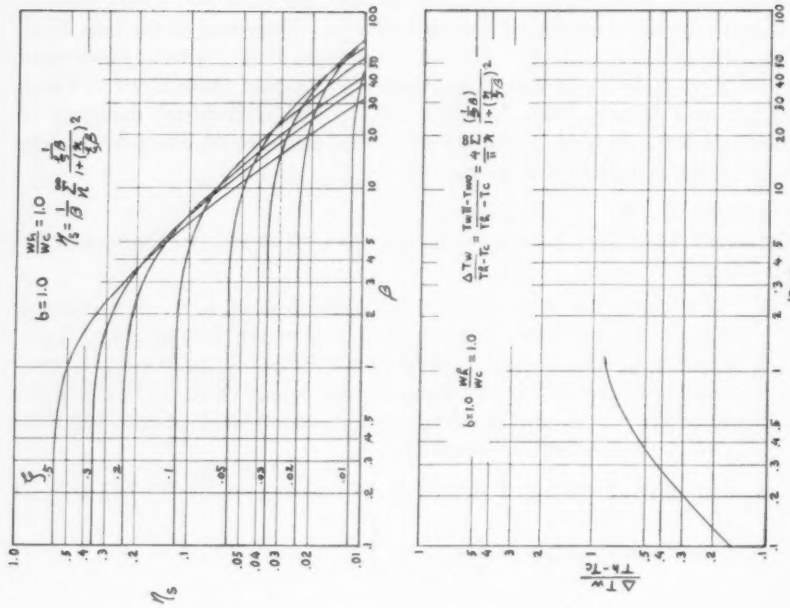

b=1.0
$\frac{W_H}{W_C}=1.0$
$\eta_s=\frac{1}{\beta}\frac{\xi\beta}{1+(\frac{\xi}{\xi\beta})^2}$
η_s
β
ξ_β
$\frac{\Delta T_w}{T_R-T_C}=\frac{T_{WH}-T_{WO}}{T_R-T_C}=\frac{4}{\pi}\frac{\xi\beta}{1+(\frac{\xi}{\xi\beta})^2}$

FIG. 10.

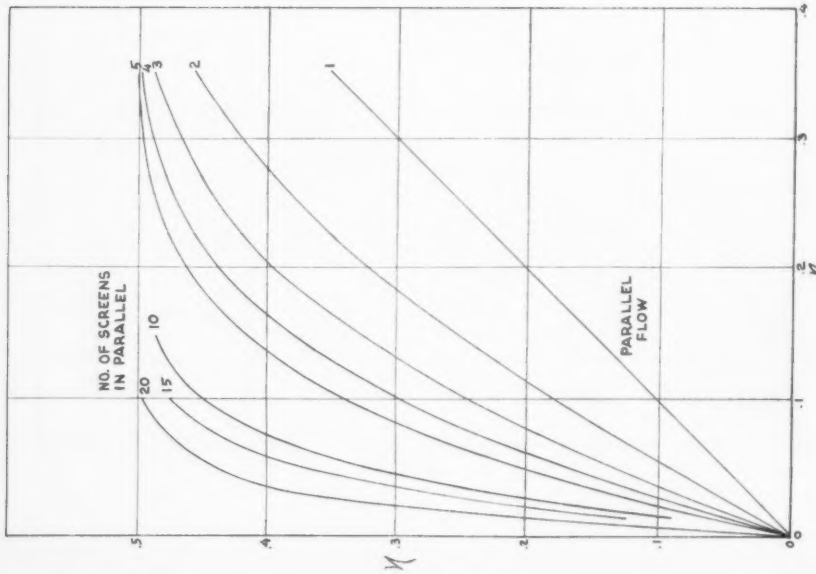

NO. OF SCREENS IN PARALLEL
20
15
10
5
4
3
2
1
PARALLEL FLOW
η_s

FIG. 9.

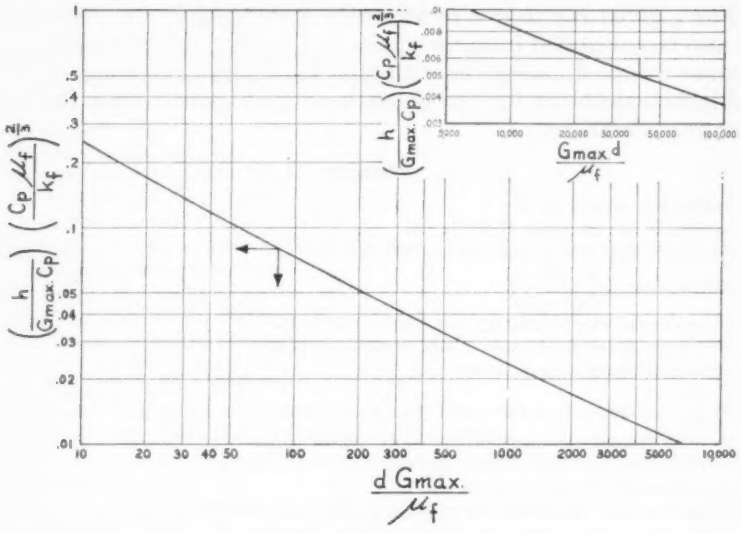


FIG. 11.

5. Calculate G from Eq. 13, N from Eq. 16.
6. Calculate free flow area A_f for either hot or cold gas from $A_f = \frac{W_h}{G_h} = \frac{W_c}{G_c}$, and calculate the outside diameter of the wire screen rotor, using

$$A_f = \frac{\pi}{4}(D_0)^2\left[1 - \left(\frac{D_i}{D_0}\right)^2\right]\left(\frac{1 - m\alpha/2\pi}{m}\right).$$

NOTE: $\left(\frac{D_i}{D_0}\right)$ may be arbitrarily selected, but is usually from 0.1 to 0.3.

7. Calculate the total wire screen core weight W_w using

$$W_w = \frac{\pi}{4}(D_0)^2\left[1 - \left(\frac{D_i}{D_0}\right)^2\right]\left(\frac{\pi}{4}\right)\left(\frac{d}{s}\right)^2 2s\rho_w n_s.$$

8. Calculate the core thickness of the wrire screen layers l, using

$$l = (n_s - 1)p + d \qquad \text{(in inches)}.$$

9. Calculate the core volume using

$$\text{Core Volume} = \frac{\pi}{4}(D_0)^2\left[1 - \left(\frac{D_i}{D_0}\right)^2\right]\left(\frac{l}{12}\right).$$

For a given rotating wire screen heat exchanger, the performance can also be calculated from the charts in a similar, but reverse, manner. In other words, knowing the values of V_a, N, etc., the value of η may readily be calculated for the given heat exchanger under the given operating conditions.

SYMBOLS

A_f = free flow area in sq. ft.
C_p = specific heat of gas in BTU/lb. °R.
C_{p_w} = specific heat of wire material in BTU/lb. °R.
c = Constant $= \left(\frac{4h}{\rho_w C_{p_w} d}\right)$
D_i = inside diameter of rotating wire screen rotor in ft.
D_0 = outside diameter of rotating wire screen rotor in ft.
d = wire diameter in ft. or inches
f = $\frac{W_h}{W_c}$
G = mass flow velocity per unit area in lb./sec. ft.^{2} = ρV
G_{max} = maximum mass flow velocity per unit area in lbs./sec. ft.^{2}
g = acceleration of gravity in ft./sec.^{2}
h = heat transfer film coefficient between gas and wire in BTU/sec. ft.^{2} °R.
k = thermal conductivity of the wire in BTU/sec. ft.^{2} °R./ft.
l = thickness of the rotating wire screen rotor in ft. or inches (thickness of multi-layer screen part only)
M = Mach number $= \left(\frac{V_a}{\sqrt{\gamma g R t_s}}\right)$
m = total number of hot and cold sections in duct (see Fig. 12)

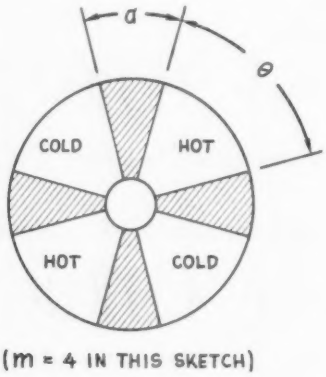


FIG. 12.

N = revolutions of rotating wire screen rotor per second
n = integral number = n^{th} harmonic in a Fourier Series
n_s = number of screens in parallel in a rotating wire screen rotor
P = total pressure in lb./in.^{2} or lb./ft.^{2}
P_r = Prandtl number
R = gas constant in ft./°R. = 53.3 (for air)

S = Transverse pitch between wires in ft. or inches
T = total temperature in °R.
T_{h_0} = total temperature of hot gas leaving the wire in °R.
T_w = instantaneous wire element temperature in °R.
ΔT_w = wire temperature rise in °R. during passage of the wire through the hot section $= T_{w_\pi} - T_{w_0}$
t = time in seconds
V_a = axial velocity of gas in ft./sec.
W = weight flow of gas in lb./sec.
W_w = total wire weight in lbs.
α = angle in degrees or radians which prevents mixing of hot and cold gases during passage through rotating wire screen rotor
$\beta = \frac{1}{2}\left(\frac{\rho C_p}{\rho_w C_{p_w}}\right)\left(\frac{1}{d/s}\right)\left(\frac{V_a}{Nd}\right)\left(\frac{1 - m\alpha/2\pi}{m}\right)$
γ = ratio of specific heats of gases
$\zeta = \frac{8}{\pi}\left(\frac{h}{G_{max} C_p}\right)\left[\frac{d/s}{(1 - d/s)^2}\right]$
η = over-all temperature effectiveness of rotating wire screen heat exchanger
η_s = temperature effectiveness of an individual wire screen
$\eta_1, \eta_2, \eta_3, \cdots \eta_n$ = temperature effectiveness of one, two, three, four . . . n wire screens in parallel
$\eta_{s_1}, \eta_{s_2}, \eta_{s_3}, \cdots \eta_{s_n}$ = temperature effectiveness of first, second, third, fourth . . . . n^{th} individual wire screen
θ = angle in degrees or radians in hot or cold section of the duct
ρ = density of gas in lb./ft.^{3}
σ = screen solidity $= 2\left(\frac{d}{s}\right) - \left(\frac{d}{s}\right)^2$
ω = equivalent angular frequency in radians per second $= \frac{mN\pi}{1 - m\alpha/2\pi}$

General Subscripts

c indicates cold gas
h indicates hot gas
i indicates initial
n indicates n^{th} harmonic in a Fourier Series
o indicates steady state time wire element entering hot section
s indicates single wire screen
w indicates wire or wire screen
π or $b\pi$ indicates steady state time wire element leaving hot section

THE MINIMAL TIME REGULATOR PROBLEM FOR LINEAR SAMPLED-DATA SYSTEMS: GENERAL THEORY*

BY

C. A. DESOER[1] AND J. WING[1]

ABSTRACT

This paper considers a single-input, linear, time-invariant, sampled-data system described by its state transition equation $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + f_{k+1}\mathbf{a}$, where the control f_{k+1} is subject to the admissibility condition $|f_{k+1}| \leqslant 1$. Necessary and sufficient conditions for controllability with admissible controls are derived. For any system satisfying these conditions, the following problem is solved: determine a scalar valued function $f(\mathbf{x})$ such that if, at each sampling instant, $f(\mathbf{x})$ is used as a control when the system is at state $\mathbf{x}$ at that sampling instant, then, given any initial state $\mathbf{x}_0$, the system will be brought to the origin in the minimum number of sampling periods. In other words, $f(\mathbf{x})$ constitutes an optimal strategy for the minimal time regulator problem. The function $f(\mathbf{x})$ is describable as follows: (1) in the state space, a hypersurface $\mathcal{C}$ is constructed; (2) the line through $\mathbf{x}$, parallel to $\mathbf{r}_1 = -\mathbf{A}^{-1}\mathbf{a}$ intersects $\mathcal{C}$ at point $\mathbf{c}$, thus $\mathbf{x} = \mathbf{c} + \lambda\mathbf{r}_1$; (3) $f(\mathbf{x}) = \text{sat}\,\lambda$.

I. INTRODUCTION

In recent years considerable attention has been given to the problem of optimal time control for linear continuous systems (1–9).[2] The most important aspect of the problem is that the control signals are restricted to belong to a closed and bounded set. In the case of a single control signal f, the restriction is that $|f(t)| \leq \gamma$, where γ is a prescribed number. The fact that f is bounded plays a pre-eminent role in the design of single-input optimal systems. The analysis of optimal control of linear discrete systems has been the object of much less attention (10, 11, 12). For discrete systems the additional restriction on control signals is that the values of the signals can change values only at fixed instants of time. The present paper solves completely the single-input minimal time regulator problem which was considered in special cases in two previous publications: in (11) a second-order case was considered and in (12), the n^{th}-order case was restricted to transfer functions that have real, distinct, non-positive poles. It is complete in the sense that no restrictions are imposed on the location or multiplicity of the poles of the transfer function other than those necessary and sufficient conditions for the system to be controllable (13).

* This research was jointly sponsored by the Electronics Research Directorate of the AF Cambridge Research Center, Air Research and Development Command, under Contract AF 19(604)–5460 and the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract AF 18(1600)–1521.

[1] Department of Electrical Engineering, University of California, Berkeley, Calif.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

This paper considers a sampled-data control system described by its state transition equation $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + f_{k+1}\mathbf{a}$ where $\mathbf{A}$ and $\mathbf{a}$ are constant $n \times n$ and $n \times 1$ matrices, respectively. Such an equation describes, for example, a system consisting of a sampler followed by a hold circuit whose output controls a black box with a transfer function $G(s)$ which is a constant divided by a polynomial in s. A constant coefficient state transition equation such as the one above may describe much more general systems, for example, under certain conditions, delay lines or transportations lags are allowable.

The minimal time regulator problem may be stated as follows: given any arbitrary initial state, it is required to bring the system to equilibrium in the minimum time with a control function f constrained by the condition $|f| \leq 1$. It is furthermore required to find the optimal strategy that will generate such an optimal control for all possible initial states.

The problem is precisely stated in Section II. In Section III, the general necessary and sufficient conditions for controllability with admissible controls are stated. Sections IV, V, and VI define and establish properties of some sets R_N' required to arrive at the proposed optimal strategy which is described in Section VII. Finally, in Section VIII, it is shown that, as $T \to 0$, the proposed optimal strategy becomes identical with the switching surface technique for generating the optimal control in the continuous case. In this paper we concentrate on establishing carefully the general theoretical framework. Since, in the general case, the optimal strategy will necessitate measurements analogous to those required in the special cases treated in (11 and 12), no description of possible implementation of the optimal strategy is included in the present paper because such descriptions are found in (11 and 12).

II. STATEMENT OF THE PROBLEM

We consider a single-input, linear, time-invariant, sampled-data system. Let $\mathbf{x}_k$ $(k = 0, 1, 2, \cdots)$ be the state of the system at time kT, where T is the sampling period. The input which will, henceforth, be referred to as the control is required to be constant during each sampling period. The control has the value f_k for $(k-1)\,T < t < kT$, $k = 1, 2, \cdots$. The system's state transition equation is

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + f_{k+1}\mathbf{a} \qquad k = 0, 1, 2, \cdots \tag{1}$$

where $\mathbf{A}$ is an $n \times n$ constant matrix, $\mathbf{a}$ is a constant column vector and $\mathbf{x}_k$ is an n-rowed column vector. Since in many applications such a system results from the sampling of a continuous system, the matrix $\mathbf{A} = e^{\mathbf{B}T}$ where $\mathbf{B}$ is the matrix appearing in the differential equation describing the continuous system. Since for any constant matrix $\mathbf{B}$,

$\det (e^{\mathbf{B}T}) \neq 0$, we assume

$$\det \mathbf{A} \neq 0. \tag{2}$$

In order to represent the common phenomenon of saturation, we assume throughout that

$$|f_k| \leq 1 \qquad \text{for all } k. \tag{3}$$

All controls satisfying (3) are called *admissible controls.* The problem we propose to solve is referred to as the *minimum time regulator problem* and is the following: given an arbitrary initial state $\mathbf{x}_0$, find an *admissible* control (specified by $f_1, f_2, \cdots$) which will bring the system to equilibrium (that is, the state $\mathbf{x} = \mathbf{0}$) in the minimum number of sampling periods. Any such admissible control is called an *optimal control.*

It is also required to find for every possible $\mathbf{x}_0$ a method for generating an *optimal control.* More precisely, it is required to find a scalar valued function of the state $\mathbf{x}$, $f(\mathbf{x})$, such that: (*a*) $|f(x)| \leq 1$ for all states $\mathbf{x}$; (*b*) if at each sampling instant, $f(\mathbf{x})$ is used as a control when the system is at state $\mathbf{x}$ at that sampling instant, then given any initial state $\mathbf{x}_0$, the system will be brought to the origin in the minimum number of sampling periods. The function $f(\mathbf{x})$ is called an *optimal strategy* because it constitutes a rule for generating optimal controls irrespective of what the initial state $\mathbf{x}_0$ might be.

III. PRELIMINARY ANALYSIS AND CONTROLLABILITY

Since no assumptions have been made concerning $\mathbf{A}$ and $\mathbf{a}$ (outside the fact that they are constant and (2)), we must first determine the conditions under which the problem has a solution. Suppose the control $f_1, f_2, \cdots, f_N$ were applied, then

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 + f_1\mathbf{a} \\
\mathbf{x}_2 &= \mathbf{A}^2\mathbf{x}_0 + f_1\mathbf{A}\mathbf{a} + f_2\mathbf{a} \\
&\vdots \\
\mathbf{x}_N &= \mathbf{A}^N\mathbf{x}_0 + f_1\mathbf{A}^{N-1}\mathbf{a} + f_2\mathbf{A}^{N-2}\mathbf{a} + \cdots + f_N\mathbf{a}.
\end{aligned}$$

Now let

$$\mathbf{r}_k = -\mathbf{A}^{-k}\mathbf{a} \qquad k = 1, 2, \cdots \tag{4}$$

and we have thus established the fact that the system can be brought to the origin in N sampling periods (that is, $\mathbf{x}_N = \mathbf{0}$) if and only if

$$\mathbf{x}_0 = f_1\mathbf{r}_1 + f_2\mathbf{r}_2 + \cdots + f_N\mathbf{r}_N. \tag{5}$$

Let us first consider the problem when the control signal f_k is not required to be admissible, that, is, it is not subject to inequality (3). We introduce now a concept due to Kalman (13).

Definition: The system (1) is *controllable* if an input can be found with the following property: given any initial state $\mathbf{x}_0$, the system can be brought to the origin $\mathbf{0}$ in a finite number of sampling periods.

Theorem 1 (13). The system (1) is controllable if and only if $\mathbf{a}$, $\mathbf{Aa}$, $\mathbf{A}^2\mathbf{a}$, $\cdots$, $\mathbf{A}^{n-1}\mathbf{a}$ are linearly independent.

Proof: From (5), $\mathbf{x}_0$ can be brought to the origin in n sampling periods if and only if $\mathbf{x}_0$ can be written as

$$\mathbf{x}_0 = f_1\mathbf{r}_1 + f_2\mathbf{r}_2 + \cdots + f_n\mathbf{r}_n.$$

Therefore, for any $\mathbf{x}_0$, a control $\{f_1, f_2, \cdots, f_n\}$ exists with that property if and only if $\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n\}$ constitute a basis for the state space; in other words if and only if $\mathbf{A}^{-1}\mathbf{a}$, $\mathbf{A}^{-2}\mathbf{a}$, $\cdots$, $\mathbf{A}^{-n}\mathbf{a}$ are linearly independent. Since $\det \mathbf{A} \neq 0$, this is equivalent to $\mathbf{a}$, $\mathbf{Aa}$, $\cdots$, $\mathbf{A}^{n-1}\mathbf{a}$ being linearly independent.

Finally, if the system cannot be brought to the origin in n sampling periods, this cannot be done in any finite number of sampling periods. This is a consequence of the Cayley Hamilton theorem (14) which states that $\mathbf{A}^n$ is a linear combination of $\mathbf{I}$, $\mathbf{A}$, $\cdots$, $\mathbf{A}^{n-1}$; therefore $\mathbf{A}^n\mathbf{a}$ is a linear combination of $\mathbf{a}$, $\mathbf{Aa}$, $\cdots$, $\mathbf{A}^{n-1}\mathbf{a}$. By iteration, it follows that $\mathbf{A}^k\mathbf{a}$ $(k \geq n)$ has the same property. Consequently if the system cannot be brought to the origin from some $\mathbf{x}_0$ in n sampling periods, it is impossible to do so in any finite number of sampling periods. This completes the proof of Theorem 1.

In the above result, the controls were not required to be admissible. We consider next the effects of this additional restriction.

Definition: A system is said to be *controllable by admissible controls* if for any initial state $\mathbf{x}_0$, there is a finite length *admissible* control $f_1, f_2, \cdots, f_N$ which brings $\mathbf{x}_0$ to the origin $\mathbf{0}$.

Observe that the duration of the admissible control, N, depends on $\mathbf{x}_0$ and it is not bounded by a duration M independent of $\mathbf{x}_0$ as it is in the case where the controls are not required to be admissible.

Theorem 2: The system (1) is *controllable by admissible controls*, if and only if

(*a*) $\mathbf{a}$, $\mathbf{A}^1\mathbf{a}$, $\mathbf{A}^2\mathbf{a}$, $\cdots$, $\mathbf{A}^{(n-1)}\mathbf{a}$ are linearly independent,
(*b*) the eigenvalues of $\mathbf{A}$ are in the closed unit circle, (that is, $|\lambda_k| \leq 1$).

Proof: Let us prove first, necessity. We assume that the system is controllable by admissible controls. Hence, it is controllable and (*a*)

holds by Theorem 1. Suppose now that (*b*) does not hold, that is, a particular eigenvalue is outside the circle, say $|\lambda_k| > 1$. By assumption, with N sufficiently large and the $|f_i| \leq 1$, the sum $-\sum_{i=1}^{N} f_i \mathbf{A}^{-i}\mathbf{a}$ can represent any point in the state space. Consider the fundamental formula[3] for $\mathbf{A}^{-i}$ and the resulting coefficient of $\mathbf{Z}_{k0}\mathbf{a}$ of the above sum; it turns out to be

$$\sum_{i=1}^{N} f_i \lambda_k^{-i}.$$

Now, however large N is,

$$\left|\sum_{i=1}^{N} f_i \lambda_k^{-i}\right| \leq \sum_{i=1}^{N} |f_i| \frac{1}{|\lambda_k|^i} < \sum_{i=1}^{\infty} \frac{1}{|\lambda_k|^i} = \frac{1}{|\lambda_k| - 1}.$$

Therefore, no admissible control exists which will make the coefficient of $\mathbf{Z}_{k0}\mathbf{a}$ larger than $1/|\lambda_k| - 1$. Hence, for example, if $\mathbf{x}_0 = \xi \mathbf{Z}_{k0}\mathbf{a}$ with $\xi > 1/|\lambda_k| - 1$, $\mathbf{x}_0$ is not controllable. This contradicts the assumption, therefore, no eigenvalue may be outside the unit circle and (*b*) is established.

Let us next prove sufficiency: we assume that (*a*) and (*b*) hold and establish controllability by admissible controls. Since (*a*) holds and since $\det \mathbf{A} \neq 0$, all the sets of vectors $(\mathbf{r}_{pn+1}, \mathbf{r}_{pn+2}, \cdots, \mathbf{r}_{pn+n})$, for $p = 0, 1, 2, \cdots$, form a basis for the state space. Given any $\mathbf{x}_0$ we define $\mathbf{x}_0^{(p)}$, for $p = 0, 1, \cdots$, as follows: expand $\mathbf{x}_0$ with respect to the p^{th} basis, thus

$$\mathbf{x}_0 = \sum_{i=1}^{n} \eta_i^{(p)} \mathbf{r}_{pn+i}.$$

For each p, divide each $\eta_i^{(p)}$ by $\max_{1 \leq i \leq n} |\eta_i^{(p)}|$; let the result be $\xi_i^{(p)}$; by construction $|\xi_i^{(p)}| \leq 1$. The vector $\mathbf{x}_0^{(p)}$ is defined by

$$\mathbf{x}_0^{(p)} = \sum_{i=1}^{n} \xi_i^{(p)} \mathbf{r}_{pn+i}. \tag{6}$$

The vector $\mathbf{x}_0^{(p)}$ lies in the direction of $\mathbf{x}_0$ and has positive (>0) length. Clearly $\mathbf{x}_0$ is controllable by admissible controls in at least Nn sampling periods if $\sum_{p=1}^{N-1} \mathbf{x}_0^{(p)}$ has a smaller length than $\mathbf{x}_0$ and $\sum_{p=1}^{N} \mathbf{x}_0^{(p)}$ has a length greater or equal to $\mathbf{x}_0$. It remains to show that given any $\mathbf{x}_0$ such a finite N exists. The only way in which no such finite N would exist is for $\mathbf{x}_0^{(p)} \to 0$ as $p \to \infty$. This can only happen if $\eta_i^{(p)} \to \infty$ as $p \to \infty$ for some $1 \leq i \leq n$. Considering the expression for $\eta_i^{(p)}$ given

[3] See Appendix I.

by Cramér's rule in solving (6), this can only happen if

$$\det (\mathbf{r}_{pn+1}, \mathbf{r}_{pn+2}, \cdots, \mathbf{r}_{pn+n}) \to 0 \qquad \text{as} \qquad p \to \infty .$$

However, since $\mathbf{r}_{pn+i} = \mathbf{A}^{-pn}\mathbf{r}_i$,

$$\det (\mathbf{r}_{pn+1}, \cdots, \mathbf{r}_{pn+n}) = (\det \mathbf{A})^{-pn} \det (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n).$$

Now $\det \mathbf{A} = \prod_{k=1}^{s} \lambda_k{}^{m_k}$ where m_k is the multiplicity of the eigenvalue λ_k. From (*b*), $|\det \mathbf{A}| \leq 1$ hence $\mathbf{x}_0{}^{(p)}$ cannot go to zero as $p \to \infty$ and the existence of N is established. Q.E.D.

Since the problem stated in Section II is meaningful only if the system is *controllable by admissible controls*, henceforth we make the following

$$\left.\begin{array}{l} \textit{Assumption:}\ \text{The matrix } \mathbf{A} \text{ and the vector } \mathbf{a} \text{ are such that} \\ (a)\ \mathbf{a}, \mathbf{Aa}, \mathbf{A}^2\mathbf{a}, \cdots, \mathbf{A}^{n-1}\mathbf{a} \text{ are linearly independent} \\ (b)\ \text{the eigenvalues of } \mathbf{A} \text{ are in the closed unit circle (that is, } |\lambda_k| \leq 1). \end{array}\right\} \tag{7}$$

A direct consequence of this assumption and of Eq. 4 is that: for any integer k, the n vectors $\mathbf{r}_k, \mathbf{r}_{k+1}, \cdots, \mathbf{r}_{k+n-1}$ are linearly independent.

IV. THE SETS R_N' AND R_N

It is obvious that since we are interested in generating for each initial state an optimal control the sets R_N', to be defined presently, will play a leading role. By definition, for any integer N,

$$\left.\begin{array}{l} R_N' \text{ is the set of all initial states that can be brought to the origin} \\ \text{by an admissible control in } N \text{ sampling periods or less.} \end{array}\right\} \tag{8}$$

It follows immediately that R_N' defined by (8) is characterized by

Property 1: (1) R_N' is also the set of all states that can be written as a sum

$\eta_1\mathbf{r}_1 + \cdots + \eta_N\mathbf{r}_N$ where $|\eta_i| \leq 1$ for $i = 1, 2, \cdots, N$; symbolically,

$$R_N' = \{\mathbf{x} | \mathbf{x} = \sum_{i=1}^{N} \eta_i\mathbf{r}_i,\ |\eta_i| \leq 1 \text{ for } i = 1, 2, \cdots, N\}.$$

(2) R_N' is convex, that is, if $\mathbf{x}_1 \in R_N'$ and $\mathbf{x}_2 \in R_N'$, then $\mathbf{x} = \lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in R_N'$ for all $\lambda \in [0, 1]$.

Suppose, for example, that $n = 3$ and that $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ are as shown in Fig. 1, then R_1' is the segment A_1A_2 and R_2' is the parallelogram $B_1B_2B_3B_4$ shown in Fig. 2; R_3' is the parallelepiped $C_1C_2, \cdots, C_8$

shown in Fig. 3; R_4' is the prism $D_1D_2, \cdots, D_{12}$ shown in Fig. 4. Consider another example: $n = 2$ and the eigenvalues of $\mathbf{A}$ are $\exp(\pm j\alpha)$ where $\alpha = 53°$; the vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_8$ are shown in Fig. 5; the sets $R_1', R_2', \cdots, R_8'$ are shown in Fig. 6.

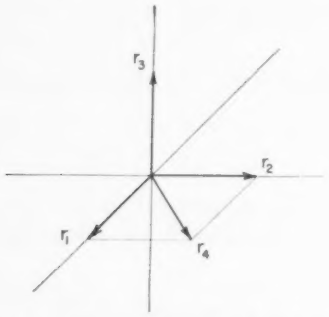


FIG. 1. The vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ of the example illustrated in Figs. 2–4.

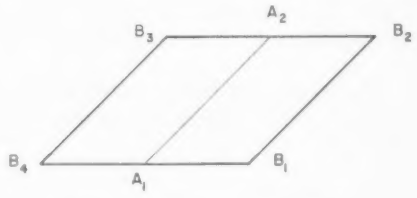


FIG. 2. The parallelogram R_2'.

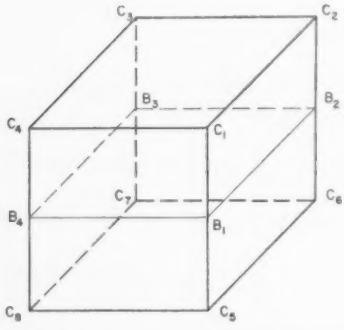


FIG. 3. The parallelepiped R_3'.

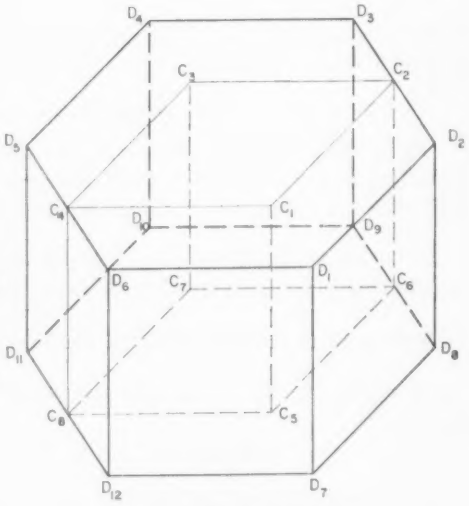


FIG. 4. The polyhedron R_4'.

For future use we have to introduce the idea of a *convex hull* (**18**). By definition, the convex hull of a finite set of points $S = \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p\}$ is the smallest convex set containing S; or equivalently, $\mathbf{y} = \sum_{i=1}^{p} \lambda_i \mathbf{y}_i$ is a point of the convex hull of S if and only if (*a*) $\lambda_i \geq 0$ and

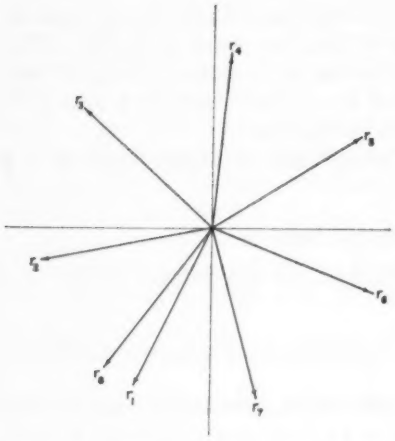


FIG. 5. The vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_8$ for the example illustrated in Fig. 6.

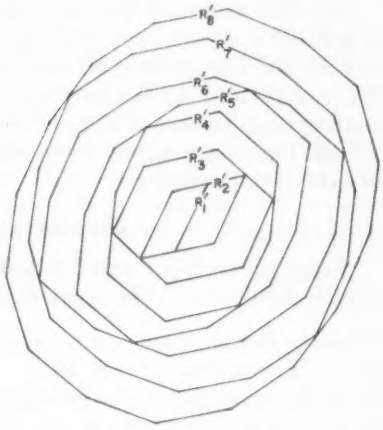


FIG. 6. The regions $R_1', R_2', \cdots, R_8'$.

(*b*) $\sum_{i=1}^{p} \lambda_i = 1$. Physically the convex hull of S, denoted by $[S]$, is the set of all points $\mathbf{y}$ which are centers of gravity of the points $\mathbf{y}_i$ of S when $\mathbf{y}_i$ is assigned a mass λ_i.

Let, now, V_N be the set of 2^N points given by $\sum_{i=1}^{N} \epsilon_i \mathbf{r}_i$ where the ϵ_i can take either the value -1 or 1. We then have

Property 2:[4] R_N' is the convex hull of V_N; symbolically, $R_N' = [V_N]$. This property is interesting because it specifies that R_N' is a polyhedron and that its vertices are points of the set V_N, although it does not specify which of these are vertices and which are interior points of R_N'. Now for control purposes, in addition to R_N' we are interested in R_N, defined as follows:

R_N is the set of initial states that can be brought to the origin by an admissible control in N sampling periods and no less. R_0 is the origin $\{\mathbf{0}\}$ itself.

Clearly from the definitions, $R_N' = \bigcup_{k=0}^{N} R_k$, that is, the union of $R_0, R_1, R_2, \cdots, R_N$. Also $R_N = R_N' - R'_{N-1}$, that is, R_N is obtained from R_N' by deleting from it all points of R'_{N-1}. A useful relation between R_N and R'_{N-1} is given by

Property 3:[4] If $\mathbf{x}_0 \in R_N$, then $\mathbf{x}_0 = \mathbf{p} + \eta_N \mathbf{r}_N$ where $\mathbf{p}$ is a boundary point of R'_{N-1} and $|\eta_N| \leq 1$.

This property calls for two remarks:

[4] The proof of this property is in Appendix II.

(1) The converse to property 3 is not necessarily true, that is, if[5] $\mathbf{p} \in \partial R'_{N-1}$ and $|\eta_N| \leq 1$, $\mathbf{x} = \mathbf{p} + \eta_N \mathbf{r}_N$ may not be in R_N. This may be the case either because the vector $\eta_N \mathbf{r}_N$ points (from $\mathbf{p}$) into R'_{N-1} or because $\mathbf{r}_N$ lies in the face of R'_{N-1} that contains $\mathbf{p}$ and η_N is small enough so that $\mathbf{p} + \eta_N \mathbf{r}_N$ is also in that face.

(2) There are some cases for which the representation $\mathbf{x}_0 = \mathbf{p} + \eta_N \mathbf{r}_N$ is not unique.

V. CONSTRUCTION OF R'_{N+1} FROM R_N'

From properties 1 and 2 it follows that, for any integer N, R_N' is a convex polyhedron. In particular:

R_1' is the line segment, centered on the origin, joining $\mathbf{r}_1$ to $-\mathbf{r}_1$, that is, $R_1' = \{\mathbf{x} | \mathbf{x} = f_1 \mathbf{r}_1, |f_1| \leq 1\}$.
R_2' is the parallelogram, centered on the origin, whose edges are parallel to $\mathbf{r}_1$ and $\mathbf{r}_2$; its vertices are $-\mathbf{r}_1 - \mathbf{r}_2$, $-\mathbf{r}_1 + \mathbf{r}_2$, $\mathbf{r}_1 + \mathbf{r}_2$, $\mathbf{r}_1 - \mathbf{r}_2$. Symbolically, $R_2' = \{\mathbf{x} | \mathbf{x} = f_1 \mathbf{r}_1 + f_2 \mathbf{r}_2, |f_1| \leq 1, |f_2| \leq 1\}$.
$\vdots$
R_n' is the hyperparallelepiped, centered on the origin, whose edges are parallel to $\mathbf{r}_1, \cdots, \mathbf{r}_n$ and whose 2^n vertices are of the form $\sum_1^n \epsilon_i \mathbf{r}_i$, where the sequences (ϵ_i) constitute the 2^n possible sequences of ones and minus ones.

For $N > n$, R_N' is obtained inductively. Suppose R_N' is known, let us construct R'_{N+1}. R_N', being a convex polyhedron, is completely defined by its vertices (see property 2). Let us classify the vertices of R_N' in two classes: in the following δ is an arbitrarily small positive number. The class V_N^+ contains all the vertices $\mathbf{p}_i$ of R_N', such that $\mathbf{p}_i + \delta \mathbf{r}_{N+1} \notin R_N'$. The class V_N^- contains all the vertices $\mathbf{q}_i$ of R_N', such that $\mathbf{q}_i - \delta \mathbf{r}_{N+1} \notin R_N'$. These two classes usually have common vertices, see Fig. 4, the vertices C_2, C_4, C_6, C_8. It follows from the fact that, for all N, R_N' is a convex polyhedron and from properties 2 and 3 that we have

Property 4: The vertices of R'_{N+1} are obtained from those of R_N' as follows:

(*a*) to each vertex $\mathbf{p}_i$ of V_N^+ corresponds the vertex of R'_{N+1}: $\mathbf{p}_i + \mathbf{r}_{N+1}$
(*b*) to each vertex $\mathbf{q}_i$ of V_N^- corresponds the vertex of R'_{N+1}: $\mathbf{q}_i - \mathbf{r}_{N+1}$.

This property follows directly from properties 1, 2, and 3. Let us, now, comment on the geometrical aspects of the relation between

[5] We use the standard notation of $\partial R_N'$ to indicate the boundary of R_N', that is, the set of all points $\mathbf{x}$, every neighborhood of which contains at least one point of R_N' and one point of the complement of R_N'.

R'_{N+1} and R_N'. With respect to the direction $\mathbf{r}_{N+1}$ (*a*) R_N' has a "top," namely, those faces that have all their vertices in V_N^+, (*b*) R_N' has a "bottom," namely, those faces that have all their vertices in V_N^-. $\partial R'_{N+1}$, the boundary of R'_{N+1}, is obtained by raising the "top" of R_N' by $\mathbf{r}_{N+1}$ and lowering the "bottom" of R_N' by $-\mathbf{r}_{N+1}$.

As an illustration consider the case shown in Figs. 1–4. By considering Figs. 1, 3, and 4, it follows that V_3^+ consists of C_1, C_2, C_6, C_5, C_8, and C_4; V_3^- consists of C_2, C_6, C_7, C_8, C_4, C_3. Note that the vertices C_2, C_6, C_4, C_8 of R_3' belong to V_3^+ and V_3^-. This example is a particular case since $\mathbf{r}_4$ is parallel to the faces C_1, C_2, C_3, C_4 and C_5, C_6, C_7, C_8 of R_3'. As a result the face D_1, D_2, D_3, D_4, D_5, D_6 of R_4' contains the face C_1, C_2, C_3, C_4 of R_3'.

It is intuitively obvious that

Property 5:[6] For each N, R_N' has edges parallel to $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$.

VI. DEFINITION AND CONSTRUCTION OF THE CRITICAL SURFACE

The critical surface $\mathfrak{C}$ is an $(n-1)$-dimensional hypersurface which will play a crucial role in obtaining the optimal control. We shall construct it step by step in a process to be described presently.

Let us consider a sequence of sets of points: ρ_1 contains only the origin. ρ_2 is the segment joining $-\mathbf{r}_2$ to $\mathbf{r}_2$, that is, $\rho_2 = \{\mathbf{x} | \mathbf{x} = \eta_2 \mathbf{r}_2, |\eta_2| \leq 1\} \cdots \rho_n$ is an $(n-1)$-dimensional parallelepiped defined by

$$\rho_n = \{\mathbf{x} | \mathbf{x} = \sum_{i=2}^{n} \eta_i \mathbf{r}_i, \ |\eta_i| \leq 1, \ i = 2, 3, \cdots, n\}.$$

Let us note that since the vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ are linearly independent, the sets $\rho_1, \rho_2, \cdots, \rho_n$ lie in the $(n-1)$-dimensional subspace spanned by $\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_n$.

For any $N \geq n$, we define ρ_{N+1} in terms of ρ_N and R_N' inductively. Consider the intersection of ρ_N with the boundary of R_N', that is, the set $\rho_N \cap \partial R_N'$. Since R_N' is a convex polyhedron this intersection consists of sets of points that lie on faces of R_N'. Let these faces be denoted by $\Phi_{N,1}, \Phi_{N,2}, \cdots$. To construct ρ_{N+1} we consider each face $\Phi_{N,i}$ and (1) if $\mathbf{r}_{N+1}$ is not parallel to any vector lying in the face $\Phi_{N,i}$, we construct the set $\Psi_{N+1,i}$ defined by

$$\Psi_{N+1,i} = \{\mathbf{x} | \mathbf{x} = \mathbf{y} + \lambda \mathbf{r}_{N+1} \text{ where } \mathbf{y} \in \Phi_{N,i} \cap \rho_N \text{ and } \lambda \text{ belong either to } (0, 1] \text{ or } [-1, 0) \text{ the choice being made so that } \mathbf{x} \notin R_N'\} \quad (9)$$

(2) if $\mathbf{r}_{N+1}$ is parallel to some vector of the face $\Phi_{N,i}$, then $\Psi_{N+1,i} = \rho_N \cap \Phi_{N,i}$. Note that in case 2, $\Psi_{N+1,i} \subset \rho_N$. By definition,

$$\rho_{N+1} \equiv \bigcup_i \Psi_{N+1,i}. \quad (10)$$

---

[6] This property is proved in Appendix II.

This construction is illustrated by considering again the example of Figs. 1–4. Figure 7 shows the parallelepiped R_3' and ρ_3 which is the parallelogram F_1, F_2, F_3, F_4. The faces Φ_{31}, Φ_{32}, Φ_{33}, and Φ_{34} are indicated. Consider now Fig. 8. Since $\rho_3 \cap \Phi_{31}$ is the line segment F_1, F_4

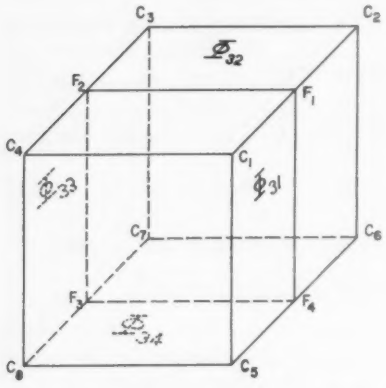


FIG. 7. The parallelepiped $C_1, \cdots, C_8$ is R_3' of the example of Figs. 1–4.

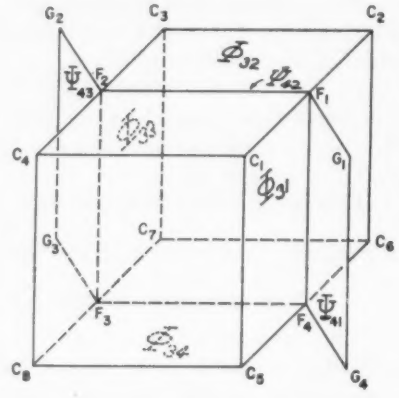


FIG. 8. Illustration of the construction of ρ_4 from ρ_3.

and since $\mathbf{r}_4$ is not parallel to any vector of the face Φ_{31}, the parallelogram Ψ_{41} is shown as F_1, F_4, G_1, G_4. Similarly for the face Φ_{33} the parallelogram Ψ_{43} is shown as F_2, F_3, G_2, G_3. Now, since $\mathbf{r}_4$ is parallel to a vector of the top and bottom faces of R_3', Ψ_{42} is the segment F_1, F_2 and Ψ_{44} is the segment F_3, F_4. ρ_3 consists of two parallelograms F_1, F_4, G_1, G_4; F_2, F_3, G_2, G_3 and two line segments F_1, F_2 and F_3, F_4. Reference to Fig. 6 of (12) shows that ρ_4 consists of the parallelograms p_{24}^+, p_{34}^+, p_{24}^-, p_{34}^-, ρ_5 of the parallelograms p_{25}^+, p_{35}^+, p_{45}^+, p_{25}^-, p_{35}^- and p_{45}^-.

Before defining the critical surface, let us add a few remarks on the sets ρ_N. (*a*) Since ρ_N is an $(n-1)$-dimensional set, the points $\mathbf{y} \in \rho_N \cap \partial R_N'$ form an $(n-2)$-dimensional set. Therefore, in case 1, the set $\Psi_{N+1,i}$, which is obtained by sweeping all points $\mathbf{y} \in \rho_N \cap \Phi_{N,i}$ in an outward direction along the vector $\mathbf{r}_{N+1}$, is an $(n-1)$-dimensional polygon. In case 2, $\Psi_{N+1,i}$ is an $(n-2)$-dimensional polygon. In either case, the polygons $\Psi_{N+1,i}$ are connected to ρ_N. In either case, also, the $\Psi_{N+1,i}$ start on a face of R_N' and terminate on a face of R'_{N+1}. ρ_{N+1}, the union of all the $\Psi_{N+1,i}$, is a set of polygons that join faces of R_N' to faces of R'_{N+1}. (*b*) Observe that $\rho_N \subset R_N$.

Definition of the Critical Surface $\mathcal{C}$: $\mathcal{C} \equiv \bigcup_{k=1}^{\infty} \rho_k$

A useful property for the derivation of the optimal control is

Property 6:[7] For each integer N, the critical surface $\mathcal{C}$ intersects the boundary of R_N' only on faces containing edges parallel to $\mathbf{r}_1$.

This allows us to say that ρ_{N+1} is the union of polygons starting on faces of R_N' containing edges parallel to $\mathbf{r}_1$ and terminating on faces of R'_{N+1} containing edges parallel to $\mathbf{r}_1$. See also Fig. 8.

As a final step we state

Property 7:[7] Any straight line parallel to $\mathbf{r}_1$ intersects the critical surface at one and only one point.

Let us stress that property 7 states that for any such straight line an intersection point exists and that it is unique.

Referring again to the example of Figs. 1–4, we note that the portion of the critical surface contained in R_4' is the two-dimensional surface G_2, F_2, F_1, G_1, G_4, F_4, F_3, G_3 shown in Fig. 8. In the example of Figs. 5 and 6, the critical surface is one-dimensional, hence is a polygonal line. The portion of it that lies in R_8' is shown in Fig. 9. It is easy to observe that in both cases properties 6 and 7 are satisfied.

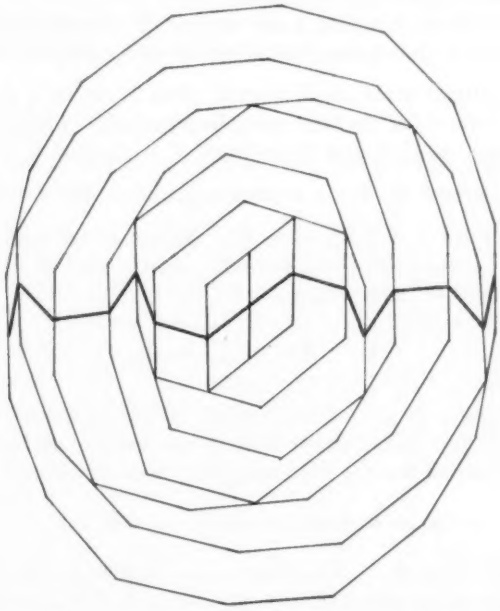


FIG. 9. This illustration refers to the example already considered in Figs. 5 and 6: the heavily drawn polygonal line is the portion of the critical surface $\mathcal{C}$ (which is one-dimensional in this case) which lies in R_8'.

[7] The proof is in Appendix II.

VII. THE PROPOSED OPTIMAL STRATEGY

We can now reformulate the definition of optimal control (given in Section II) in terms of the concepts R_N' and R_N. Suppose the initial state $\mathbf{x}_0 \in R_N$, then the admissible control f_1 is *optimal* if and only if the state $\mathbf{x}_1 \in R'_{N-1}$, where $\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 + f_1\mathbf{a}$. Observe that $\mathbf{x}_0 \in R_N$ and $\mathbf{x}_1 \in R'_{N-1}$ imply that $\mathbf{x}_1 \in R_{N-1}$. An *optimal strategy* is a rule which for any state $\mathbf{x}$ gives an optimal control for that $\mathbf{x}$; it is a function whose domain is the state space and whose range is the set of admissible controls. Any computer that generates at each sampling instant an optimal control is simply a realization of an optimal strategy. Conversely the first step in the design of such a computer is to find an optimal strategy.

First let us introduce a

Definition of the Critical Slabs ζ

The set ζ consists of all points $\mathbf{x}$ such that $\mathbf{x} = \mathbf{c} + \lambda\mathbf{r}_1$ where $\mathbf{c}$ is a point of $\mathcal{C}$ and $|\lambda| \leq 1$. Symbolically,

$$\zeta = \{\mathbf{x} | \mathbf{x} = \mathbf{c} + \lambda\mathbf{r}_1, \qquad \mathbf{c} \in \mathcal{C}, \qquad |\lambda| \leq 1\}. \tag{11}$$

Note that for a given $\mathbf{x}$, $\mathbf{c}$ and λ are unique, by property 7.

We shall derive the optimal strategy in successive steps:

(1) If the initial state $\mathbf{x}_0$ is on the critical surface $\mathcal{C}$, then $f_1 = 0$ is an optimal control for the first sampling period. This is immediately established: since $\mathbf{x}_0 \in \mathcal{C}$ and there is an N such that $\mathbf{x}_0 \in R_N$, we have $\mathbf{x}_0 \in R_N \cap \mathcal{C}$, hence $\mathbf{x}_0$ has a representation of the form $\mathbf{x}_0 = \sum_{i=2}^{N} \eta_i\mathbf{r}_i$ where $|\eta_i| \leq 1$ for $i = 2, 3, \cdots, N$. Suppose we apply the control $f_1 = 0$ during the first sampling period. At the end of it, the system is in state $\mathbf{x}_1$ and by (1) and (4),

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = \sum_{i=1}^{N-1} \eta_{i+1}\mathbf{r}_i.$$

Hence, by property 1, $\mathbf{x}_1 \in R'_{N-1}$ and $f_1 = 0$ is optimal.

(2) Suppose the initial state $\mathbf{x}_0$ is in the critical slab ζ, then $f_1 = \lambda$ is an optimal control for the first sampling period, where λ is defined by

$$\mathbf{x}_0 = \mathbf{c} + \lambda\mathbf{r}_1 \qquad \text{with} \qquad \mathbf{c} \in \mathcal{C}. \tag{12}$$

Let N be such that $\mathbf{x}_0 \in R_N$. Since ρ_N, hence ζ, intersects $\partial R_N'$ on faces containing edges parallel to $\mathbf{r}_1$ and since R_N' is convex, the point $\mathbf{c}$ defined by (12) is also in R_N', hence

$$\mathbf{x}_0 = \lambda\mathbf{r}_1 + \sum_{i=2}^{N} \eta_i\mathbf{r}_i$$

where $|\eta_i| \leq 1$ for $i = 2, 3, \cdots, N$. Since $\mathbf{x}_0 \in \zeta$, $|\lambda| \leq 1$ and $f_1 = \lambda$ is an admissible control. Again using (1) and (4) we have

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 + \lambda\mathbf{b} = \sum_{i=1}^{N-1} \eta_{i+1}\mathbf{r}_i$$

where $|\eta_{i+1}| \leq 1$ for $i = 1, 2, \cdots, N-1$. Hence $\mathbf{x}_1 \in R'_{N-1}$ and $f_1 = \lambda$ is optimal.

(3) Suppose $\mathbf{x}_0$ does not belong to the critical slab ζ, then $f_1 = \operatorname{sgn} \lambda$ is an optimal control for the first sampling period, where λ is defined by (12).

Let N be such that $\mathbf{x}_0 \in R_N$. Since $\mathbf{x}_0 \notin \zeta$ and $\mathbf{x}_0 \in R_N'$ the $\mathbf{c}$ and λ defined by (12) are such that $\mathbf{c} \in R_N'$ and $|\lambda| > 1$. Therefore the lemma of Appendix III applies and

$$\mathbf{x}_0 = (\operatorname{sgn} \lambda)\,\mathbf{r}_1 + \sum_{i=2}^{N} \gamma_i \mathbf{r}_i$$

with $|\gamma_i| \leq 1$, $i = 2, \cdots, N$. Therefore, as above, $f_1 = \operatorname{sgn} \lambda$ is both admissible and optimal.

These three results allow us to formulate the

Proposed Optimal Stragegy

Suppose that at the present sampling instant the system is in state $\mathbf{x}_0$. Compute λ such that

$$\mathbf{x}_0 = \mathbf{c} + \lambda\mathbf{r}_1$$

where $\mathbf{c} \in \mathfrak{C}$. Then $f_1 = \operatorname{sat} \lambda$ is an optimal control for the next sampling period.[8] The function sat λ is the function $f(\mathbf{x})$ mentioned in Section II: it is the *optimal strategy*.

Let us note that the above stated optimal strategy is not the only optimal strategy. For an illustration of the nonuniqueness of the optimal strategy see (11), Fig. 10 in particular.[9] The proposed strategy seems to be the simplest and also the one that will give naturally the optimal strategy for the continuous case as $T \to 0$.

Let us note also that the proposed optimal strategy requires only the simulation of the critical surface $\mathfrak{C}$ and the determination of λ, the distance along the direction $\mathbf{r}_1$, of the present state $\mathbf{x}$ to the critical surface $\mathfrak{C}$. For special cases, examples are to be found in (11 and 12).

---

[8] sat $x = x$ for $|x| \leq 1$ and sat $x = \operatorname{sgn} x$ elsewhere.

[9] Another optimal strategy would then be as follows: at $\mathbf{x}_0 \epsilon R_N$, take $f_1 = \lambda'$, where λ' is the λ smallest in absolute value such that $\mathbf{A}\mathbf{x}_0 + f_1\mathbf{a} \epsilon R'_{N-1}$.

VIII. RELATION BETWEEN THE CONTINUOUS CASE AND THE DISCRETE CASE

In this section we propose to discuss heuristically the relation between the optimal control of a continuous system and that of the corresponding sampled system. More specifically, we consider a *fixed* continuous system

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + f(t)\mathbf{b} \tag{13}$$

where $\mathbf{B}$ and $\mathbf{b}$ are, respectively, $n \times n$ and $n \times 1$ constant matrices and $|f| \leq 1$. Associated to the continuous system (13), we consider the discrete system obtained by sampling the control $f(t)$ and keeping it constant during the sampling period: thus, for the sampled system the control is $f_k = f[(k-1)T]$ for the interval $(k-1)T < t < kT$. The resulting discrete system is characterized by

$$\mathbf{x}_{k+1} = \mathbf{A}(T)\mathbf{x}_k + f_{k+1}\mathbf{a}(T) \tag{14}$$

where the dependency of $\mathbf{A}$ and $\mathbf{a}$ on T is explicitly indicated. The following relations will be useful.

$$\mathbf{A} = e^{\mathbf{B}T} = \mathbf{I} + T\mathbf{B} + \frac{T^2}{2!}\mathbf{B}^2 + \cdots \tag{15}$$

$$\mathbf{a} = \int_0^T e^{-\mathbf{B}\tau}\mathbf{b}\,d\tau = T\mathbf{b} - \frac{T^2}{2}\mathbf{B}\mathbf{b} + \frac{T^3}{3!}\mathbf{B}^2\mathbf{b} - \cdots. \tag{16}$$

It is known (7) that, for the continuous system, if $\mathbf{b}, \mathbf{B}\mathbf{b}, \cdots, \mathbf{B}^{n-1}\mathbf{b}$ are linearly independent and if the eigenvalues of $\mathbf{B}$ have negative real parts, then, for each initial state $\mathbf{x}_0$, (1) there exists an admissible minimal time control that brings the system to $\mathbf{0}$, (2) the minimal time control is piecewise constant and (3) the set of all admissible controls that simultaneously satisfy the maximum principle and bring the system to $\mathbf{0}$ consists of only one element.

In order to show how the control and the optimal strategy of the discrete system merge into that of the continuous system as $T \to 0$, we consider a sequence of sampling periods: $T_l = T_0/2^l$, $l = 1, 2, \cdots$ and T_0 is a fixed number. For each l, the proposed optimal control will bring the system to the origin starting from $\mathbf{x}_0$ in a time $\tau(\mathbf{x}_0, l)$. Clearly $\tau(\mathbf{x}_0, l+1) \leq \tau(\mathbf{x}_0, l)$ because the optimal control for T_l is still an admissible control for $T_{l+1} = T_l/2$. Therefore the sequence of $\tau(\mathbf{x}_0, l)$ forms a monotonically decreasing sequence of positive numbers, hence as $l \to \infty$, $\tau(\mathbf{x}_0, l) \to \tau(\mathbf{x}_0)$. The limit $\tau(\mathbf{x}_0)$ is equal to $\tau_c(\mathbf{x}_0)$, the minimal time required by the continuous system (13) to bring the system to $\mathbf{0}$ starting from $\mathbf{x}_0$. This follows because, for sufficiently large l, a piecewise constant (over intervals of length T_l) approximation of the optimal control of (13) will be an admissible control of (14) and

the corresponding solution of (14) is an ϵ-approximation (16) to the solution of (13). Therefore, as $l \to \infty$, this procedure will give a solution of (14) which will require a time arbitrarily close to $\tau_c(\mathbf{x}_0)$. Therefore $\tau(\mathbf{x}_0)$ cannot be larger than $\tau_c(\mathbf{x}_0)$. Conversely, it is impossible to have $\tau(\mathbf{x}_0) < \tau_c(\mathbf{x}_0)$ because the optimal control of (14), for any l, is an admissible control of (13). Hence $\tau(\mathbf{x}_0) = \tau_c(\mathbf{x}_0)$.

Consider now what happens to the critical surface (which we shall denote by $\mathcal{C}(T_l)$ to emphasize its dependence on T_l) as $l \to \infty$. As $l \to \infty$, the critical surface $\mathcal{C}(T_l)$ becomes a smooth surface which we shall denote by $\mathcal{C}(0)$. We justify heuristically this assertion by the following observations: (1) the length of the edges of the polygons that make up $\mathcal{C}(T_l)$ go to zero as $l \to \infty$, because $\mathbf{a} \to 0$ as seen from (16); (2) the angle between successive $\mathbf{r}_k$'s $\to 0$ as $l \to \infty$. This can be checked by showing that the ratio of $(\mathbf{r}_{k+1} - \mathbf{r}_k)/\|\mathbf{r}_k\| \to 0$ as $l \to \infty$, even if k is increased so that kT_l remains constant. The only case where the normal to the tangent plane of $\mathcal{C}(0)$ will not be a continuous function of position is at points where, in order to construct ρ_{N+1}, the sign of $\mathbf{r}_{N+1}$ had to be reversed with respect to the sign of $\mathbf{r}_N$ used to construct ρ_N at that point.

Since $\mathbf{r}_1 \to 0$ as $l \to \infty$, the limiting behavior of the optimal strategy is clear: the sign of the forcing function is reversed as soon as the state space point reaches $\mathcal{C}(0)$. In other words $\mathcal{C}(0)$ becomes a switching surface. Thus we see that, as the sampling period approaches 0, the proposed optimal strategy merges into the well-known switching surface technique which is used to generate the optimal control for continuous systems.

IX. CONCLUSION

This paper presented a complete derivation of an optimal strategy for the minimal time regulator problem for discrete systems described by equations of the form (1) and for which assumptions 2 and 7 hold. It has been shown that unless these assumptions are satisfied, the problem does not make sense. The derivation is general in the sense that only the assumptions 2 and 7 are imposed on the system (1). These assumptions restrict the eigenvalues of $\mathbf{A}$ to have their magnitude positive and not greater than unity. The procedures described in the paper are applicable to all such cases irrespective of whether the eigenvalues are distinct or multiple, real or occur in complex conjugate pairs. The proposed optimal strategy defines for every point of the state space an optimal control, in other words, it defines a function $f(\mathbf{x})$ for all states $\mathbf{x}$ such that $f(\mathbf{x})$ is an optimal control. From this point of view it is closely related to the concept of switching surfaces used in the continuous case (see discussion in Section VIII) and does not require consideration of the usually exponentially increasing solutions of the adjoint system introduced in (**6** and **7**).

APPENDIX I: THE CANONICAL EXPRESSION OF A^{-i}

The purpose of this appendix is to state briefly the fundamental formula (17).

Let $\mathbf{A}$ be a matrix with $\psi(\lambda)$ as the minimal polynomial and $\lambda_1, \lambda_2, \cdots, \lambda_s$ as eigenvalues. Expand $\frac{1}{\psi(\lambda)}$ in partial fraction and collect terms

$$\frac{1}{\psi(\lambda)} = \sum_{k=1}^{s} \sum_{l=0}^{m_k-1} \frac{c_{kl}}{(\lambda - \lambda_k)^{l+1}} = \sum_{k=1}^{s} \frac{n_k(\lambda)}{(\lambda - \lambda_k)^{m_k}}$$

where m_k is the multiplicity of λ_k as a root of $\psi(\lambda)$. Define the matrices

$$\mathbf{Z}_{kl} = (\mathbf{A} - \lambda_k \mathbf{I})^l \phi_k(\mathbf{A})$$

where

$$\phi_k(\lambda) = \psi(\lambda)[n_k(\lambda)/(\lambda - \lambda_k)^{m_k}].$$

Now it can be shown that if $f(\lambda)$ is analytic in an open set containing $\lambda_1, \lambda_2, \cdots, \lambda_s$ then

$$f(\mathbf{A}) = \sum_{k=1}^{s} \sum_{l=0}^{m_k-1} f(\lambda_k{}^{(l)}) \mathbf{Z}_{kl}.$$

In particular, for any integer i,

$$\mathbf{A}^{-i} = \sum_{k=1}^{s} \sum_{l=0}^{m_k-1} \frac{d^l}{d\lambda^l} (\lambda^{-i}) \Big|_{\lambda=\lambda_k} \mathbf{Z}_{kl}.$$

This is the fundamental formula for $\mathbf{A}^{-i}$ used in the proof of Theorem 2.

APPENDIX II: PROOF OF THE PROPERTIES 1–7

The purpose of this appendix is to give proofs for the properties given in the body of the paper.

Property 1 follows immediately from the definitions.

Property 2: R_N' is the convex hull of V_N; symbolically $R_N' = [V_N]$.

Proof: In order to show that $R_N' = [V_N]$ we first show that (1) $[V_N] \subset R_N'$ and (2) $R_N' \subset [V_N]$. The desired equality follows then from the definition of the equality of two sets.

(*a*) It is immediate that $R_1' = [V_1]$ and that, for any N, $[V_N] \subset R_N'$. The latter follows from that any point of V_N, that is, $\sum_{i=1}^{N} \epsilon_i \mathbf{r}_i$, is in R_N' by property 1.

(*b*) The property will be established if the following induction is proved: $[V_{N-1}] \supset R'_{N-1}$ implies $[V_N] \supset R_N'$. Let $\mathbf{x}$ be any point of R_N' therefore $\mathbf{x} = \sum_{i=1}^{N} \eta_i \mathbf{r}_i$ with $|\eta_i| \leq 1$. Now $\sum_{i=1}^{N-1} \eta_i \mathbf{r}_i \in R'_{N-1}$, hence by the induction assumption

$$\sum_{i=1}^{N-1} \eta_i \mathbf{r}_i = \sum_{k=1}^{2^{N-1}} \lambda_k \mathbf{p}_k$$

where $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_{2^{N-1}}$ are the points of V_{N-1}, $\lambda_k \geq 0$, $\sum_{k=1}^{2^{N-1}} \lambda_k = 1$. Observe that $\mathbf{p}_k \pm \mathbf{r}_N$ is a point of V_N, hence

$$\mathbf{x} = \sum_{i=1}^{N-1} \eta_i \mathbf{r}_i + \eta_N \mathbf{r}_N = \sum_{k=1}^{2^{N-1}} \lambda_k \mathbf{p}_k + \eta_N \mathbf{r}_N$$

$$= \left(\frac{1}{2} + \frac{\eta_N}{2}\right) \sum_{k=1}^{2^{N-1}} \lambda_k (\mathbf{p}_k + \mathbf{r}_N) + \left(\frac{1}{2} - \frac{\eta_N}{2}\right) \sum_{k=1}^{2^{N-1}} \lambda_k (\mathbf{p}_k - \mathbf{r}_N).$$

This last equality shows that $\mathbf{x}$ is a point of $[V_N]$. Hence, the induction is established, hence $[V_N] \supset R_N'$ for all N. This together with the result of (a) proves the theorem.

Property 3: If $\mathbf{x}_0 \in R_N$, then $\mathbf{x}_0 = \eta_N \mathbf{r}_N + \mathbf{p}$ where $\mathbf{p}$ is a boundary point of R'_{N-1} and $|\eta_N| \leq 1$.

Proof: Since $\mathbf{x}_0 \in R_N$, $\mathbf{x}_0$ has a representation of the form

$$\mathbf{x}_0 = \sum_{i=1}^{N} \eta_i \mathbf{r}_i \qquad \text{with} \qquad |\eta_i| \leq 1.$$

Since such a representation is not necessarily unique, consider all possible such representations and, in particular, the one that has the smallest η_N in absolute value. Let that representation be

$$\mathbf{x}_0 = \sum_{i=1}^{N} \eta_i' \mathbf{r}_i.$$

Clearly we have $|\eta_i'| \leq 1$, $i = 1, 2, \cdots, N$. Obviously, $\eta_N' \neq 0$ for otherwise $\mathbf{x}_0$ would be in R'_{N-1} which is impossible since it is in R_N. Also

$$\mathbf{x}_0 - \eta_N' \mathbf{r}_N = \sum_{i=1}^{N-1} \eta_i \mathbf{r}_i \in R'_{N-1}.$$

The point $\sum_{i=1}^{N-1} \eta_i \mathbf{r}_i$ is a boundary point of R'_{N-1}: indeed, let η_N'' be obtained by reducing η_N' by ϵ in absolute value. The point $\mathbf{x}_0 - \eta_N'' \mathbf{r}_N$ is not in R'_{N-1} by definition of η_N' but is arbitrarily close to the point $\sum_{i=1}^{N-1} \eta_i \mathbf{r}_i$ of R'_{N-1}. Therefore, the latter is a boundary point (15) of R'_{N-1}.

Property 4 is established in two steps: (a) $R'_{N+1} \supset$ the convex hull of the vertices $\mathbf{p}_i + \mathbf{r}_{N+1}$ and $\mathbf{q}_i - \mathbf{r}_{N+1}$ and (b) this same convex hull includes R'_{N+1}. The proof is immediate if one uses properties 1, 2, and 3.

Property 5: For every integer N, R_N' has edges parallel to $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$.

Proof: The property is true for $N = 1$. Let us establish the induction: suppose it is true for N, prove that it is true for $N + 1$. By property 4, each edge of R_N' parallel to $\mathbf{r}_i$ is translated either by $\mathbf{r}_{N+1}$ or $-\mathbf{r}_{N+1}$ to become an edge of R'_{N+1}, therefore R'_{N+1} has at least one edge parallel to $\mathbf{r}_i$ for $i = 1, 2, \cdots, N$. If $\mathbf{r}_{N+1}$ is parallel to one of these $\mathbf{r}_i$'s the property is established. But if $\mathbf{r}_{N+1}$ is not parallel to any of the $\mathbf{r}_i$'s ($i = 1, 2, \cdots, N$) then V_N^+ and V_N^- have at least one point, say $\mathbf{q}$, in common. Let by definition $\mathbf{q} \in V_N^+ \cap V_N^-$, then any $\delta > 0$ however small, $\mathbf{q} - \delta \mathbf{r}_{N+1} \notin R_N'$ and $\mathbf{q} + \delta \mathbf{r}_{N+1} \notin R_N'$. Therefore to this $\mathbf{q}$ will correspond an edge of R'_{N+1} parallel to $\mathbf{r}_{N+1}$, namely the edge that joins $\mathbf{q} + \mathbf{r}_{N+1}$ to $\mathbf{q} - \mathbf{r}_{N+1}$. This establishes the induction and property 5 is established.

Property 6: For any integer N, the critical surface $\mathcal{C}$ intersects the boundary of R_N' only on faces containing edges parallel to $\mathbf{r}_1$. Observe that the intersection of $\mathcal{C}$ with $\partial R_N'$ is equal to that of ρ_N with $\partial R_N'$. For $1 \leq N \leq n$, the property is easily verified since from the above description of the ρ_N and property 1, we have

$$R_N' = \{\mathbf{x} | \mathbf{x} = \mathbf{y} + \lambda \mathbf{r}_1, \quad |\lambda| \leq 1, \quad \mathbf{y} \in \rho_N\} \qquad \text{for} \qquad 1 \leq N \leq n.$$

For $N \geq n$ we use induction: we assume that for some $N \geq n$, ρ_N intersects $\partial R_N'$ on faces containing edges parallel to $\mathbf{r}_1$ and we have to prove that this is still true for ρ_{N+1}.

Let $\Phi_{N,i}$ be a face of R_N' containing points of ρ_N. By the induction assumption $\Phi_{N,i}$ has an edge parallel to $\mathbf{r}_1$. Now case 1: $\mathbf{r}_{N+1}$ is not parallel to any vector of $\Phi_{N,i}$ (that is, $\mathbf{r}_{N+1}$ cannot be expressed as a linear combination of the edges of the face $\Phi_{N,i}$). Then all points of $\Phi_{N,i}$ undergo a translation of either $\mathbf{r}_{N+1}$ or $-\mathbf{r}_{N+1}$ to become a face of R'_{N+1}. This new face

will therefore also have some edges parallel to $\mathbf{r}_1$. Now, the points $\mathbf{y} \in \Phi_{N,i} \cap \rho_N$ will undergo the same translation to form the polygon $\Psi_{N+1,i}$ and the intersection of ρ_{N+1} with $\partial R'_{N+1}$ will include points on the new face obtained by translation from $\Phi_{N,i}$. Therefore ρ_{N+1} intersects $\partial R'_{N+1}$ on faces that contain edges parallel to $\mathbf{r}_1$ and the induction is established for case 1. In case 2, $\mathbf{r}_{N+1}$ is parallel to some vector of the face $\Phi_{N,i}$. After translation by $\mathbf{r}_{N+1}$ in an outward direction the new face of R'_{N+1} includes $\Phi_{N,i}$ and ρ_{N+1} intersects this new face since $\Psi_{N+1,i}$ has only points on this face. The induction is thus also established in case 2.

Property 7: Any straight line parallel to $\mathbf{r}_1$ intersects the critical surface $\mathcal{C}$ at one and only one point.

Proof: Let l be the arbitrary straight line, more precisely, let

$$l = \{\mathbf{x} | \mathbf{x} = \mathbf{x}_0 + \mu \mathbf{r}_1, \quad \mu \in (-\infty, \infty), \quad \mathbf{x}_0 \text{ arbitrary but fixed}\}.$$

(*a*) Let us first show that l cannot intersect ρ_N at two distinct points $\mathbf{a}$ and $\mathbf{b}$.

For $1 \leq N \leq n$, this is impossible because ρ_N is a set in a subspace spanned by $(\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N)$ and, by assumption, $\mathbf{r}_1$ is linearly independent of the set of vectors $(\mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_n)$. Therefore the property follows for $N \leq n$.

Let us now use induction to prove it for $N > n$. More specifically we assume that the property holds for some $N > n$ and wish to prove that it holds for $N + 1$. We establish the induction by contradiction: suppose l intersects ρ_{N+1} at two distinct points $\mathbf{a}$ and $\mathbf{b}$. Clearly if $\mathbf{r}_{N+1}$ is parallel to $\mathbf{r}_1$, reference to the construction of ρ_{N+1} from ρ_N shows that $\rho_{N+1} \subset \rho_N$ in this case; hence we have a direct contradiction with the induction assumption. It remains, therefore, to establish the induction for the case where $\mathbf{r}_{N+1}$ is not parallel to $\mathbf{r}_1$. With an obvious notation,

$$\mathbf{a} - \mathbf{b} = (\mu_a - \mu_b)\mathbf{r}_1.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are in ρ_{N+1}, by construction of ρ_{N+1}, there are two points $\mathbf{y}_a$ and $\mathbf{y}_b$ such that

$$\begin{aligned} \mathbf{a} &= \mathbf{y}_a + \lambda_a \mathbf{r}_{N+1} \qquad \mathbf{y}_a \in \rho_N \cap \partial R_N', \qquad |\lambda_a| \leq 1 \\ \mathbf{b} &= \mathbf{y}_b + \lambda_b \mathbf{r}_{N+1} \qquad \mathbf{y}_b \in \rho_N \cap \partial R_N', \qquad |\lambda_b| \leq 1 \end{aligned}$$

Let us project these two relations along a direction parallel to $\mathbf{r}_1$ onto an $(n-1)$-dimensional subspace. Let "$\frown$" indicate the projected quantities. We have

$$\left.\begin{aligned} \hat{\mathbf{a}} &= \hat{\mathbf{b}} \\ \hat{\mathbf{y}}_a &= \hat{\mathbf{a}} - \lambda_a \hat{\mathbf{r}}_{N+1} \\ \hat{\mathbf{y}}_b &= \hat{\mathbf{a}} - \lambda_b \hat{\mathbf{r}}_{N+1} \end{aligned}\right\} \tag{A-1}$$

and $\hat{\mathbf{r}}_{N+1} \neq 0$. Since $\mathbf{y}_a$ and $\mathbf{y}_b \in \rho_N \cap \partial R_N'$, they lie on faces of R_N' which contain edges parallel to $\mathbf{r}_1$, therefore $\hat{\mathbf{y}}_a$ and $\hat{\mathbf{y}}_b$ are on the boundary of the projection $\hat{R}_N'$. Equations A-1 mean that $\hat{\mathbf{y}}_a$ and $\hat{\mathbf{y}}_b$ are on a line originating from $\hat{\mathbf{a}}$ and whose direction is that of $\hat{\mathbf{r}}_{N+1}$. Since $\hat{R}_N'$ is convex, this line intersects $\partial \hat{R}_N'$ at two points $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$, where $\hat{\mathbf{p}}$ is the intersection point closest to $\hat{\mathbf{a}}$. Since to obtain $\mathbf{a}$ and $\mathbf{b}$ from $\mathbf{y}_a$ and $\mathbf{y}_b$ one had to move these points along $\mathbf{r}_{N+1}$ in a direction *outward* with respect to R_N', it follows that $\hat{\mathbf{y}}_a$ and $\hat{\mathbf{y}}_b$ must coincide with $\hat{\mathbf{p}}$. Consequently $\mathbf{y}_a - \mathbf{y}_b = \nu \mathbf{r}_1$ for some ν. Note that $\nu \neq 0$, for otherwise since $\mathbf{r}_{N+1}$ is not parallel to $\mathbf{r}_1$, we would also have $\mathbf{a} = \mathbf{b}$. Thus

$$\mathbf{y}_a - \mathbf{y}_b = \nu \mathbf{r}_1, \qquad \nu \neq 0$$

which implies that there is a straight line parallel to $\mathbf{r}_1$ (namely, $\mathbf{x} = \mathbf{y}_a + \lambda \mathbf{r}_1, \lambda \in (-\infty, \infty)$) which intersects ρ_N at two distinct points $\mathbf{y}_a$ and $\mathbf{y}_b$. This contradicts the induction assumption. Therefore the induction is proved and the statement (*a*) proven for all N.

(*b*) Let us next show that l cannot intersect $\mathcal{C}$ at a point $\mathbf{a}$ of ρ_N and at a point $\mathbf{b}$ of $\rho_{N'}$ where $\mathbf{b} \notin \rho_N$ and $N' \neq N$.

Let us project orthogonally all the R_k''s, ρ_k's and l on a two-dimensional plane Π perpen-

dicular to $\mathbf{r}_1$. Indicate the projected sets by "$\sim$." Now $\bar{l}$ is a single point $\bar{\mathbf{x}}_0$, $\bar{R}_k'$ is a convex polygon of the plane II. For any N, $\rho_i \subset R_N'$ for $i \leq N$ implies that $\bar{\rho}_i \subset \bar{R}_i'$ for $i \leq N$. Observe that for any N, ρ_N is the union of polygons $\Psi_{N,i}$ which start on a face of R'_{N-1} and end on a face of R_N'; each of these faces has some edges parallel to $\mathbf{r}_1$, hence project on II as a point or a line segment; each of these points or line segments is on the boundary of $\bar{R}_N'$. Therefore $\bar{\rho}_N$ is outside the interior of $\bar{R}'_{N-1}$. Since $\bar{\rho}_N \subset \bar{R}_N'$, then $\bar{\rho}_N \subset \bar{R}_N' - (\widetilde{\mathring{R}'_{N-1}})$.[10] Furthermore, reference to the method of construction of R_N' from R'_{N-1} and that of $\Psi_{N,i}$ shows that we actually have $\bar{\rho}_N = \bar{R}_N' - (\widetilde{\mathring{R}'_{N-1}})$, for all N. This fact together with the general inclusion $\cdots \subset \bar{R}'_{N-1} \subset \bar{R}_N' \subset \bar{R}'_{N+1} \subset \cdots$ shows that if $N \neq N'$, $\bar{\rho}_N$ and $\bar{\rho}_{N'}$ cannot have interior points in common (although if $|N - N'| = 1$, they may have boundary points in common). The statement (b) follows from this last fact.

(c) Any line l intersects $\mathcal{C}$: This follows from the above reasoning because the assumed controllability by admissible controls of the system implies that $\bigcup_{k=1}^{\infty} R_k'$ fills the whole state space. Therefore $\bigcup_{k=1}^{\infty} \bar{R}_k'$ fills the whole plane II. Therefore there is an N such that $\bar{\mathbf{x}}_0 \in \bar{R}_N'$ and since $\bar{R}_N' = \bigcup_{k=1}^{N} \bar{\rho}_N$, there is a $k \leq N$ such that $\bar{\mathbf{x}}_0 \in \bar{\rho}_k$.

APPENDIX III: LEMMA

Lemma: Suppose $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$ are linearly dependent. Suppose $\mathbf{y}$ has the following representations

$$\mathbf{y} = \alpha_1'\mathbf{r}_1 + \sum_{i=2}^{N} \alpha_i\mathbf{r}_i = \beta_1\mathbf{r}_1 + \sum_{j=2}^{N} \beta_j\mathbf{r}_j \tag{A-2}$$

where

$$\alpha_1' > 1 \quad \text{and} \quad |\alpha_i| \leq 1 \quad \text{for } i = 2, 3, \cdots, N \tag{A-3}$$

and

$$|\beta_j| \leq 1 \quad \text{for } j = 1, 2, \cdots, N. \tag{A-4}$$

Then $\mathbf{y}$ has a representation of the form

$$\mathbf{y} = \mathbf{r}_1 + \sum_{i=2}^{N} \gamma_i\mathbf{r}_i \quad \text{with} \quad |\gamma_i| \leq 1, \quad i = 2, 3, \cdots, N. \tag{A-5}$$

Proof: If $\beta_1 = 1$ the theorem is proved. Let us then consider the case where $\beta_1 < 1$. From (A-2) it follows that

$$(\beta_1 - \alpha_1')\mathbf{r}_1 + \sum_{i=2}^{N} (\beta_i - \alpha_i)\mathbf{r}_i = \mathbf{0}. \tag{A-6}$$

Pick the number k as

$$k = \frac{\alpha_1' - 1}{\alpha_1' - \beta_1}.$$

Observe that $0 < k < 1$ because of (A-3) and (A-4). Now multiply (A-6) by k and add it to (A-2),

$$\mathbf{y} = [\alpha_1' + k(\beta_1 - \alpha_1')]\mathbf{r}_1 + \sum_{i=2}^{N} [\alpha_i + k(\beta_i - \alpha_i)]\mathbf{r}_1.$$

Hence

$$\mathbf{y} = \mathbf{r}_1 + \sum_{i=2}^{N} \gamma_i\mathbf{r}_i$$

[10] The superscript ^{0} over the symbol of a set indicates the interior (15) of that set.

where

$$\gamma_i \equiv \alpha_i + k(\beta_i - \alpha_i) = (1 - k)\alpha_i + k\beta_i \qquad i = 2, 3, \cdots, N.$$

Since, for $i = 2, 3, \cdots, N$, both α_i and β_i belong to $[-1, 1]$ and $0 < k < 1$ then

$$|\gamma_i| \leq 1 \qquad i = 2, 3, \cdots, N$$

and representation (A-5) is established.

Corollary: Under the same assumptions and notations but for the first inequality (A-3) replaced by $\alpha_1' < -1$, the conclusion becomes

$$\mathbf{y} = -\mathbf{r}_1 + \sum_{i=2}^{N} \gamma_i \mathbf{r}_i \qquad \text{with} \qquad |\gamma_i| \leq 1 \qquad \text{for } i = 2, 3, \cdots, N.$$

REFERENCES

(1) D. C. McDONALD, "Nonlinear Techniques for Improving Servo Performance," *Proc. Natl. Electronics Conf.*, Vol. 6, pp. 400–421 (1950).

(2) A. M. HOPKIN, "A Phase Plane Approach to the Design of Saturating Servomechanism," *Trans. AIEE*, Vol. 70, Part I, pp. 631–639 (1951).

(3) D. BUSHAW, "Optimal Discontinuous Forcing Terms," in *Contribution to the Theory of Nonlinear Oscillations*, Vol. IV, edited by S. Lefshetz, Princeton, Princeton University Press, 1958, pp. 29–58.

(4) I. BOGNER AND L. F. KAZDA, "An Investigation of the Optimum Switching Criteria for Higher Order Servomechanisms," *Trans. AIEE*, Vol. 73, Part II, pp. 118–127 (1954).

(5) R. BELLMAN, I. GLICKSBERG AND O. GROSS, "On the Bang Bang Control Problem," *Q. Appl. Math.*, Vol. 14, pp. 11–18 (1956).

(6) C. A. DESOER, "The Bang Bang Control Problem Treated by Variational Techniques," *Information and Control*, Vol. 2, pp. 333–348 (1959).

(7) V. G. BOLTYANSKI, R. V. GAMKRELIDZE AND L. S. PONTRYAGIN, "The Theory of Optimal Processes I," *Izvestia Akad. Nauk SSSR Seriya Matem.*, Vol. 24, pp. 3–42 (1960).

(8) N. N. KRASOVSKII, "On the Theory of Optimum Control," *Prikl. Mat. Mekh.*, Vol. 23, No. 4, pp. 625–639 (1959).

(9) E. B. LEE, "Mathematical Aspects of the Synthesis of Linear Minimum Response-Time Controllers," *IRE Trans. on Automatic Control*, Vol. AC-5, No. 4, pp. 283–289 (1960).

(10) R. E. KALMAN, "Optimal Nonlinear Control of Saturating Systems by Intermittent Action," *1957 IRE WESCON Convention Record*, Part 4, pp. 130–135.

(11) C. A. DESOER AND J. WING, "An Optimal Strategy for a Saturating Sampled Data System," *IRE Trans. on Automatic Control*, Vol. AC-6, pp. 5–15, February, 1961.

(12) C. A. DESOER AND J. WING, "A Minimal Time Discrete System," *IRE Trans. on Automatic Control*, Vol. AC-6, pp. 111–125, May, 1961.

(13) R. E. KALMAN, "On the General Theory of Control Systems," *Proc. First International Congress of Automatic Control*, Moscow, 1960.

(14) B. FRIEDMAN, "Principles and Techniques of Applied Mathematics," New York, John Wiley & Sons, 1956.

(15) T. M. APOSTOL, "Mathematical Analysis, A Modern Approach to Advanced Calculus," Reading (Mass.), Addison Wesley Publishing Co., 1957.

(16) E. A. CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations," New York, McGraw-Hill Book Co., 1955.

(17) F. R. GANTMACHER, "The Theory of Matrices," New York, Chelsea Publishing Co., 1959.

(18) D. BLACKWELL AND M. A. GIRSHICK, "Theory of Games and Statistical Decisions," New York, John Wiley & Sons, 1954.

NATIONAL BUREAU OF STANDARDS NEWS

A NEW MICROCITE MACHINE

The National Bureau of Standards has recently developed an electromechanical searching device that has significant advantages for information retrieval in large collections of documents. Developed by Joshua Stern of the Bureau's staff, the machine is based on the "microcite" concept[1] and employs a special punched-card technique known as "peek-a-boo." The simplicity of the design, together with various automatic features, makes possible decentralized search facilities so that the inquiring researcher may perform the search himself.

The improved microcite machine was developed, under the sponsorship of the Department of Defense, to aid the Bureau's instrument reference service in making information on instruments and control systems more readily available to Government and industry.

The problem of effective retrieval of scientific and technological information has increased in importance and in difficulty in recent years. Much of the current work in this area is directed towards development of complex machines that will search a vast memory automatically and furnish the documents sought in a desired form. The high cost of such machines requires that their operation be centralized and inhibits direct contact of the inquirer with the search in progress. The element of feedback, which can contribute much to the effectiveness of a search, is difficult to incorporate in such centralized operation.

The importance of feedback in the search process is illustrated by the familiar game of "twenty questions." A few moments of consideration makes it clear that the great effectiveness of twenty questions in pinpointing information stems principally from the fact that each question is phrased after the answer to the previous question has been provided. This same advantage is inherent in the microcite machine. Here the researcher is at all times in control of the search process and is immediately informed of the nature of the information being discovered.

A brief description of the peek-a-boo technique and the microcite concept should aid in understanding the present development. The peek-a-boo technique uses punched cards each of which represents an index word. The positions of holes in the card identify documents pertaining to the index word. One search techique often referred to as the logical product is performed by superimposing the appropriate peak-a-

[1] "Microcite, an Aid to More Effective Referencing," *NBS Tech. News Bull.*, Vol. 41, p. 141 (1957).

boo cards and noting the positions of holes that remain unobscured by any of the selected cards. More complex logical operations can also be performed. In the ordinary use of the peek-a-boo system, the position of a hole is interpreted as a document serial number by means of a coordinate grid printed upon the card or upon a transparent overlay. The searcher is directed by these serial numbers to a file (serial list) of citations and abstracts of the documents sought or to a file of the documents themselves.

The microcite concept eliminates this latter step by immediately providing the searcher with a microfilm image of the citation and abstract. Instead of reading serial numbers, the searcher views directly at each hole position a description of a document emerging from the search. In a preliminary model, microimages of citations and abstracts were reproduced on a film overlay and read at the holes with the aid of a microscope. Such a procedure demonstrated the effectiveness of the microcite principle but would be relatively inefficient for large collections.

The new microcite is a tool that could search millions of documents. In its particular application at the Bureau, it has been designed for sets of 18,000-document peek-a-boo cards with corresponding abstracts for each set carried in a 15-in.-square area of a sheet of photographic film—the matrix. In operation, the film matrix appropriate to the peek-a-boo set being searched is selected and mounted on a drum. The operator places selected peek-a-boo cards on an illuminated area of the machine and manipulates two control wheels to position a set of cursors on each illuminated—that is, unobscured—hole in turn. Turning the control wheels causes rotation and translation of the drum carrying the film matrix, so that as the cursor is set on each hole, a full-size focused image of the citation and abstract corresponding to that hole is projected on a screen in front of the searcher.

As he proceeds, the searcher can obtain a copy of any of the abstracts he desires by moving a lever to record the desired image on a roll of photosensitive paper. Alternatively, he may note the serial numbers of the documents or may cause the serial numbers to be transcribed into punched tape or printed on paper tape. The electrically coded serial-number output that will be available for this transcription could also be fed into a document-storage machine—such as are expected to become available in the near future—which would provide copies of complete documents in response to serial-number input.

The immediate availability of the results is a powerful aid to the searcher in allowing him to guide the search effectively. He can rapidly detect and adjust for such contingencies as misinterpretation of index terms by indexer and searcher, unanticipated interference from information of whose existence he was unaware, misconceptions about the volume of relevant information in the collection, and even modification of his interests as he acquires more and more information.

POLYMER CRYSTALS WITH FOLDED CHAINS

The nature of polymer crystals is of considerable interest, since such crystals affect the mechanical, optical, and dielectric properties of these useful materials. As part of a long-range program involving the dielectric properties of polymers, scientists at the National Bureau of Standards have investigated the fundamental characteristics[2] of these materials from a theoretical point of view.

In 1958, it was discovered that certain polymers, such as polyethylene, are deposited from dilute solution in the form of flat and very thin crystalline platelets. (See, for example, the work of A. Keller and co-workers.[3,4]) The surprising feature of these plate-like crystals is that they are built up of *folded* polymer chains. The folds themselves actually form the large upper and lower surfaces of the platelets. The chain axes of the unfolded portions of the polymer molecules are roughly perpendicular to the plane of the chain folds. This is a vastly different structure from the bundle-like or sheaf-like structure that one might have expected the crystals to exhibit.

A detailed theoretical interpretation of this phenomenon has recently been presented by J. I. Lauritzen and J. D. Hoffman of the Bureau's Dielectrics Section. Their explanation involves nucleation and growth theory. They have demonstrated that in sufficiently dilute solution, the rate of formation of the seeds leading to folded crystals must exceed that of seeds leading to crystals of the bundle-like type.

The growth of each folded nucleus was then considered. It was shown that the thickness (often called the step height, l^*) will not increase rapidly as the platelet becomes larger in the x- and y-directions, thus preserving the thin platelet character of the crystal. This tendency for the step height to maintain itself during the growth process is a result of the fact that a crystal of step height l^* grows much faster than one that is either substantially thicker or thinner. The quantity l^* generally ranges from roughly 100 to 500 A, and increases with temperature of crystallization.

The theoretical study has recently been extended to the case of crystallization in bulk polymers.[5] Results were obtained which strongly suggest that substantially chain-folded crystals (lamellae) exist in bulk polymers, especially in the spherulites. Calculations have recently been made at the Bureau which give information on the smoothness of the chain-folded surfaces of the lamellae and on the details of the melting behavior to be expected for chain-folded crystals.

---

[2] For further information, see "Theory of Formation of Polymer Crystals with Folded Chains in Dilute Solution," by John I. Lauritzen, Jr., and John D. Hoffman, *NBS J. Research*, Vol. 64A (Phys. & Chem.), No. 1, p. 73 (1960).

[3] A. KELLER AND A. O'CONNOR, *Discussions, Faraday Soc.*, Vol. 25, p. 114 (1958).

[4] A. KELLER, *Phil. Mag.*, Vol. 2, p. 1171 (1957).

[5] "Theory of rate of Lamellar Spherulitic Crystallization in Bulk Polymers," by John D. Hoffman and J. I. Lauritzen, Jr., *NBS J. Research*, Vol. 65A (Phys. & Chem.), No. 4, p. 297 (1961).

THE FRANKLIN INSTITUTE

NEW BOOKS IN THE FRANKLIN INSTITUTE LIBRARY

HISTORY OF SCIENCE AND TECHNOLOGY

Newtonian Science, by E. A. Bell. E. Arnold, 1961.

A Short History of Technology from the Earliest Times to A.D. 1900, by T. K. Derry. Oxford University Press, 1961.

Rocket Development, edited by Esther C. Goddard. Prenctice-Hall, 1961.

The National Watercraft Collection, by H. I. Chapelle. Smithsonian Institution, 1960.

Discourse on Bodies in Water, by G. Galilei, translated by T. Salusbury. University of Illinois Press, 1960.

ASTRONOMY

International Space Science Symposium, edited by H. K. Bijl. Interscience, 1960.

Space Astrophysics, by W. Liller. McGraw-Hill, 1961.

Tools of the Astronomer, by G. R. Miczaika and W. M. Sinton. Harvard University Press, 1961.

Science in Space, edited by L. V. Berkner. McGraw-Hill, 1961.

MATHEMATICS

Transform Calculus for Electrical Engineers, by R. Légros. Prentice-Hall, 1961.

Concepts from Tensor Analysis and Differential Geometry, by T. Y. Thomas. Academic, 1961.

Lectures on Linear Algebra, by I. Gel'fand. Interscience, 1961.

Elements of Mathematics, by J. H. Banks. 2nd ed. Allyn and Bacon, 1961.

Partial Differential Equations and Continuum Mechanics, by R. E. Langer. University of Wisconsin Press, 1961.

Theory of Markov Processes, by E. B. Dynkin. Pergamon, 1960.

Fourier Transforms, by R. Goldberg. Cambridge University Press, 1961.

PHYSICS

Small Particle Statistics, by G. Herdan. 2nd ed. Academic, 1960.

Turbulence; Classic Papers on Statistical Theory, edited by S. K. Friedlander. Interscience, 1961.

Diffraction, Structure des Images; Influence de la Coherence de la Lumiere, by A. Marèchal. Editions de la Revue d'optique théorique et instrumentale, 1960.

Hydrometry; Theory and Practice of Hydraulic Measurements, by A. T. Troskolañski. Pergamon, 1960.

Symposium on Radiation Effects and Radiation Dosimetry. American Socity for Testing Materials, 1960.

The Fundamental Atomic Constants, by J. H. Sanders. Oxford University Press, 1961.

Electromagnetic Structure of Nucleons, by S. D. Drell. Oxford University Press, 1961.

Heat Transfer, by B. Gebhart. McGraw-Hill, 1961.

Elements of Nuclear Engineering, by G. Murphy. Wiley, 1961.

Atomic Energy Waste, by E. Glueckauf. Interscience, 1961.

Physics and Archaeology, by M. J. Aitken. Interscience, 1961.

Physics for Engineers and Scientists, by R. G. Fowler. Allyn and Bacon, 1961.

The Principles of Nuclear Magnetism, by A. Abragam. Clarendon Press, 1961.

CHEMISTRY AND CHEMICAL ENGINEERING

REINFORCED PLASTICS FOR ROCKETS AND AIRCRAFT. Symposium. American Society for Testing Materials, 1961.

INORGANIC CHEMISTRY, A GUIDE TO ADVANCED STUDY, by R. B. Heslop. Elsevier, 1960.

INTERNATIONAL SYMPOSIUM ON MACROMOLECULAR CHEMISTRY. Izdatel'stvo Akademii Nauk SSSR, 1960.

POLYMERIC MATERIALS, by C. C. Winding. McGraw-Hill, 1961.

CHEMISTRY, by M. J. Sienko. McGraw-Hill, 1961.

INTRODUCTION TO QUALITATIVE ANALYSIS, by D. C. Layde. Allyn and Bacon, 1961.

REAGENT CHEMICALS; SPECIFICATIONS, 1960. American Chemical Society, 1961.

NAME INDEX OF ORGANIC REACTIONS, by T. S. Wheeler. Interscience, 1960.

THE SCIENCE OF ADHESIVE JOINTS, by J. J. Bickerman. Academic, 1961.

SYMPOSIUM ON ADHESION AND ADHESIVES. American Society for Testing Materials, 1961.

KOLLOIDCHEMISCHES TASCHENBUCH, by A. Kuhn. Akademische Verlagsgesellschaft, 1960.

INFRARED ABSORPTION OF INORGANIC SUBSTANCES, by K. Lawson. Reinhold, 1961.

MECHANICAL ENGINEERING

CREATIVE ENGINEERING ANALYSIS, by F. L. Ryder. Prentice-Hall, 1961.

THEORIES OF ENGINEERING EXPERIMENTATION, by H. Schenck. McGraw-Hill, 1961.

THE CONSULTING ENGINEER, by C. M. Stanley. Wiley, 1961.

ENGINEERING DATA FOR PRODUCT DESIGN, by D. C. Greenwood. McGraw-Hill, 1961.

50 YEARS PROGRESS IN MANAGEMENT, 1910–1960. American Society of Mechanical Engineers, 1960.

SMALL PLANT MANAGEMENT, by W. A. MacCrehan, Jr. 2nd rev. ed. McGraw-Hill, 1960.

MECHANICAL MEASUREMENTS, by T. G. Beckwith. Addison-Wesley, 1961.

HANDBOOK OF MECHANICAL WEAR, by C. Lipson. University of Michigan Press, 1961.

PLASTIC TOOLING, by M. W. Riley. 2nd ed. Reinhold, 1961.

FRICTION AND WEAR IN MACHINERY. American Society of Mechanical Engineers, 1960.

THE ROLE OF VISCOSITY IN LUBRICATION, edited by O. C. Bridgeman. American Society of Mechanical Engineers, 1960.

DICTIONARY OF MECHANICAL ENGINEERING, by A. Del Vecchio. Philosophical Library, 1961.

THE DESIGN OF SMALL DIRECT-CURRENT MOTORS, by A. F. Puchstein. Wiley, 1961.

MECHANIZATION OF MOTION, by L. Harrisburger. Wiley, 1961.

ELECTRICAL ENGINEERING

ELECTRICAL ENGINEERING, by S. B. Hammond. McGraw-Hill, 1961.

AN INTRODUCTION TO THE PRINCIPLES OF COMMUNICATION THEORY, by J. C. Hancock. McGraw-Hill, 1961.

COMPUTER ENGINEERING, edited by S. A. Lebedev. Pergamon, 1960.

METALLURGY

SYMPOSIUM ON LOW-TEMPERATURE PROPERTIES OF HIGH-STRENGTH AIRCRAFT AND MISSILE MATERIALS. American Society for Testing Materials, 1961.

FATIGUE TESTING AND ANALYSIS OF RESULTS, by W. Weibull. Pergamon, 1961.

INTERNATIONAL SYMPOSIUM ON THE PHYSICAL CHEMISTRY OF PROCESS METALLURGY, edited by G. R. St. Pierre. Interscience, 1961.

ENGINEERED CASTINGS, by G. J. Cook. McGraw-Hill, 1961.

EXTRACTIVE METALLURGY OF COPPER, NICKEL, AND COBALT, edited by P. Queneau. Interscience, 1961.

COBALT MONOGRAPH, prepared in collaboration with the staff of Battelle Memorial Institute. Brussels, 1960.

BOOK REVIEWS

AERO-THERMODYNAMICS AND FLOW IN TURBOMACHINES, by Michael H. Vavra. 609 pages, diagrams, $5\frac{3}{4} \times 9$ in. New York, John Wiley & Sons, Inc., 1960. Price, $14.50.

This book presents a vector and dyadics analysis of fluid flow through turbomachinery. It includes applications to axial flow compressors and turbines and two-dimensional potential flow through blading cascades. A simplified three-dimensional analysis of axial turbomachines is included. Other features include an extensive appendix on vector analysis, including an extension to operational analysis involving dyadics.

The book is divided into three major parts. Part I pertains to "Fundamental Flow Relations"; Part II extends to relative and rotational motion, and also discusses isentropic and potential flows; Part III covers applications to turbomachines already mentioned.

In Part I, the author formulates the thermodynamic properties in the equations of motion, where the latter are expressed in terms of gradients of velocity and pressure. Inserting the thermodynamic property

$$T \cdot \Delta s = \nabla h - \frac{1}{e} \nabla b$$

for the pressure drops in

$$\frac{\partial V}{\partial t} + V \nabla V = -\frac{1}{e} \nabla p - F,$$

the author arrives at

$$\frac{\partial V}{\partial t} + V \nabla V = -\nabla h + T \nabla s - F,$$

where the fluid flow can be expressed in terms of enthalpy and entropy. Since the increase of entropy, Δs, is the combined effect of both external and internal heat transfers, while the irreversibility due to friction is itself $T \Delta s_R = f dF$, approximately, the advantage of this method appears only by approximating entropy increases, but retaining explicitly the friction forces f. Actually, as with steady flow, with adiabatic irreversible systems, $V \cdot \nabla V = -\nabla h$, due to the above relation. The irreversibility can likewise be accounted for by comparing actual with isentropic enthalpy changes.

The arguments relating to energy relations in the author's discussion of jet propulsion seem highly artificial, if not incorrect. In estimating the jet reaction, the velocities V_1 and V_2 can be interpreted either as relative or absolute velocities, since for the "change" in momentum, we are only concerned with the difference in velocities. However, in the statement of energy relations, it is very important to distinguish between the two.

The author (p. 75) states $H_1 = h_1 + \frac{V_1^2}{2g}$ and $H_2 = h_2 + \frac{V_2^2}{2g}$ as the total energies at stations 1 and 2, where V_1 and V_2 are interpreted as relative velocities. With the unnecessary hypothesis to prove a fundamental relation, the author assumes an "ideal" engine with enthalpies $h_1 = h_2$ and $T_1 = T_2$ and concludes $H_2 - H_1 = \frac{V_2^2}{2} - \frac{V_1^2}{2}$ as the minimum possible energy for acceleration of a particle from V_1 to V_2. It is important to point out that $\int \frac{dm_{s_2}}{2} V_2^2 - \frac{m_s V_1^2}{2} = \Delta E$, with V_1 and V_2 interpreted as *relative* velocities, is always a measure of the "mechanical" energy imparted to the flow stream, irrespective of whether the thermal cycle is reversible or irreversible. ΔE should be regarded as the *available* part of the heat input, ΔQ, for mechanical work. Hence any assumption as to the nature of the thermal cycle is unnecessary.

Actually, the total energy for mechanical work *must* include the heat energy injected into the flow stream through combustion of the fuel supply at a rate ΔQ per unit time. The heat energy retained in the flow stream at exit is measured by the exit enthalpy. With the convected enthalpies h_1 and h_2 at terminal sections, a flow rate G(lb/sec), w as

relative, and v = the velocity of plane, then ΔE is

$$\Delta E = \Delta Q + G(h_1 - h_2) = \frac{G}{2g}(w_2^2 - w_1^2)$$

which gives the relation of entrance and exit enthalpies, with heat source ΔQ. Neglecting the small mass flow of injected fuel, it can be shown that $\frac{G}{2g}(w_2^2 - w_1^2)$, that is, the square of the difference of the *relative* velocities, accounts for the heat energy transferred to mechanical work, part of which is utilized in propelling the plane and the other is lost in the slip stream. Even with the author's ideal engine, if $\Delta E = \frac{G}{2g}(w_2^2 - w_1^2) = \Delta Q$, with $h_1 = h_2$, then $T_1 = T_2$ *must be nil*. The author's ideal engine appears inconsistent with the Carnot cycle, where only a part of ΔQ is available for mechanical work, if finite exit temperatures are assumed where $h_2 > h_1$. A possible confusion arises in not clearly separating the heat energy transferred to mechanical work, and the heat energy retained in the slip stream and finally transferred to the atmosphere at its ambient temperature. This phase is concerned with the thermal efficiency. The energy transformed to mechanical work, again separates to propulsion work and to mechanical energy lost in the slip stream. This latter is related to the propulsion efficiency which the author is concerned with. The propulsion efficiency $\epsilon_p = 2(w_2 - w_1)v/(w_2^2 - w_1^2) = \frac{2v}{(w_2 + w_1)}$, where $w_1 = v$ = velocity of plane. On the other hand, the thermal efficiency is

$$\epsilon_t = \frac{\frac{G}{2g}(w_2^2 - w_1^2)}{\Delta Q}$$

and the over-all efficiency is

$$\epsilon_t \epsilon_p = \frac{G}{g}\frac{(w_2 - w_1)v}{\Delta Q}.$$

Moreover, whether the energy transferred is isentropic or irreversible, the previous equation and the above relations hold. Therefore, the author's hypothetical ideal cycle appears unwarranted for evaluating the propulsion efficiencies.

In Part II, the author discusses relative motions of the flow through rotors. The distinction between steady flow in stators and unsteady flow in rotors with respect to stator, is clearly brought out. With steady relative flow in rotor, the significance of rotational motion, the Coriolis and centrifugal component and circulation flow, are discussed. The significance of the Coriolis component and the rate of change of angular momentum should be emphasized. The characteristics of rotational flow, with vortex filaments are mathematically discussed. In this section isentropic flow of perfect gases is discussed, followed by potential flow analysis.

The remainder of the book includes applications to axial flow compressors and the analysis of cascade flow.

Special mention should be given to the author's very complete review of vector analysis, and an unique introduction to operations involving dyadics.

This book is both an interesting and valuable discussion of fluid flow analysis in turbomachinery. It extends a type of mathematics that seems indispensable to advancing the analysis and research studies of fluid flow problems.

RUPEN EKSERGIAN
The Franklin Institute Laboratories

ADVANCES IN THE ASTRONAUTICAL SCIENCES, VOL. 6, edited by Horace Jacobs and Eric Burgess. 898 pages, diagrams, illustrations, 6½ × 9¼ in. New York, The Macmillan Co., 1961. Price, $25.00.

The annual meetings of astronautical societies permit the presentation of papers which mirror both current events and thinking in the space-travel panorama. These meetings allow leaders in allied fields to explore facets of astronautics of common interest. Thus at these sessions can be heard discussions of the most important topics of the day.

In January 1960 the Sixth Annual Meeting of the American Astronautical Society was held in New York. The three-day session was the setting for 56 technical papers, three special lectures, and a space panel. This volume presents these papers.

There are nine sections to the book which

range from communications to space physics. Some of the sections, such as Guidance and Control and Space Flight Mechanics, are highly developed technical areas which need a substantial background in the subjects to understand them. Other sections are of a more popular nature.

One of the papers in the space-flight mechanics section by E. T. Benedikt concerns "Disintegration Barriers to Extremely High-Speed Space Travel." The author's thesis is that when we travel at high enough velocities, the space ship will disintegrate due to the impacts of particles we meet in space. In the reviewer's book "Target For Tomorrow" this writer showed that indeed this was a possibility and the only way to circumvent the hazard was to fabricate the interstellar vehicle from an asteroid leaving a shell several hundred feet thick. Thus while the material would wear away there would still be enough material there to maintain the integrity of the space vehicle.

Dandridge M. Cole contributed a paper on "The Feasibility of Propelling Vehicles by Contained Nuclear Explosions." This reviewer has always admired the boldness of this concept and here it is detailed for the reader. Mr. Cole in his paper is concerned with all-up weights of 175,000 tons—more than twice the weight of the largest ocean liner.

Dr. S. Fred Singer's paper, "Some Considerations of Expected Radiation Belts of Planets Mars and Venus," is another example of the analytical work being done by a ranking scientist in the space-travel field. Dr. Singer may be remembered as the scientist who in 1958 predicted and explained the radiation belts of the earth before their discovery. Here is another example of his ingenious deductions. Now it remains for us to determine experimentally whether he is right in his analysis.

These are but a few of the papers in this outstanding volume which caught the interest of this reviewer. Everyone in the space-travel field will find papers of extreme interest to him.

Advances in the Astronautical Sciences is another reference book necessary to permit one to stay abreast of this exploding science.

I. M. LEVITT
The Fels Planetarium

ELECTROMAGNETIC FIELDS AND WAVES, by R. V. Langmuir. 227 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $9.75.

Professor Langmuir states in this new book that "In such a rapidly moving field as modern electrical engineering one can scarcely do much more than train the student in the mathematical methods and basic physics of the field . . . to meet and solve problems that are not even dreamed of now." He expects the students to come quite well prepared in vector analysis, partial differential equations, Fourier series, etc. Thus, his new text is rather short, because little time is spent tutorially in building mathematical background or in discussing current practices.

What is covered is treated in a precise, elegant, and sophisticated way. It is safe to say that the quality of the book rests partly with the author's good judgment of what to leave out. Of course there is no new material; the entire text, having only 200 pages, certainly had to be prepared with strict discrimination. For example, the treatment of waveguides does not contain data on the numerous guide configurations and irises. However, the exposition is sufficiently powerful to arm the student with the means for analyzing a great many of the microwave engineering problems which may be encountered.

Advanced subjects such as plasma physics, about which engineers of the future will need to know much, are not covered at all. On the other hand, entire chapters are devoted to the fundamental Laplace and Maxwell equations. The basic mathematical techniques for dealing with dielectric phenomena and magnetic materials are similarly treated.

In summary, it is apparent that, through the courses he has taught at the California Institute of Technology, Professor Langmuir has distilled certain basic elements from such sources as Smythe, Stratton, Slater, Frank, *et al.* In teaching this highly mathematical subject the author makes a point of emphasizing the tangible reality of terms by the use of some delightful, homespun examples such as "1 newton . . . about the gravitational force on a small apple." The result is an excellent, compact and concentrated version

of the classic texts for effective use by the senior electrical engineering student.

C. W. HARGENS
The Franklin Institute Laboratories

MEN OF SPACE—VOLUME II, by Shirley Thomas. Chilton Press, 1961. Price, $3.95.

For Volume II of her series on the men who are shaping our destinies in space, Shirley Thomas has chosen ten more distinguished scientists.

While Volume I profiled the obvious first choices, with the names of the men who figured most prominently in the news, this volume is by no means a collection of biographical sketches of the second team. We must face up to the situation that in this country the news media can promote and glamorize an individual who may say and do no more or less than a close associate or colleague. We never hear of that colleague, yet he is an important contributor to space science. Thus this volume is essentially one detailing the contributions of the lesser known but still brilliant stars in the space firmament.

Some of the men have obvious connections with space as in the case of Dr. Dornberger, Dr. Whipple, Captain Truax, Dr. Lovelace, etc. Others, such as Drs. Ramo, Teller, though dynamic figures, are involved only indirectly in the space-flight picture.

The author has a unique ability of portraying the subject in vignettes which spotlight significant events. Thus the reader is permitted to fill in the obvious details. By writing in such fashion, the 225-page book really represents a much more complete volume. She has the happy faculty of portraying the scientist not only as a worker in his discipline but as a well-rounded human being who has a family and is subject to all the problems and situations which arise from this state. Biographies make fascinating reading and the author has the enviable faculty of making her subjects come alive.

The significance of these *Men in Space* profiles will be appreciated later when their pioneering work becomes history. Our children will be better off for knowing just how this adventure started and who gave it the greatest push and how they did it.

Miss Thomas has again produced a live, stimulating and highly readable book. I look forward to the next in the series.

I. M. LEVITT
The Fels Planetarium

SPACE ASTROPHYSICS, edited by William Liller. 272 pages, diagrams, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $10.00.

During the 1959–1960 year the Astronomy Department of the University of Michigan presented a series of lectures on those aspects of astronomy and astrophysics which could be investigated only from outside the atmosphere of the earth.

This volume is a compilation of the lectures given by ranking authorities at the invitation of the University. Their contributions are in most cases of original material and find their way into print for the first time.

The book is divided into two sections with the first section involving the experiments performed and to be performed in space, and the second section is concerned with the theory. The second section is an excellent "wrap-up" of our knowledge of the solar wind, corona, the electric universe and the theory behind some of the instruments which may be used to further our knowledge of space.

The second section makes it a worth-while addition to the library of the working space scientists.

I. M. LEVITT
The Fels Planetarium

BOOK NOTES

PROGRESS IN BIOPHYSICS AND BIOPHYSICAL CHEMISTRY, VOL. 11, edited by J. A. V. Butler, B. Katz and R. E. Zirkle. 255 pages, diagrams, plates, $6\frac{1}{2} \times 10$ in. London, Pergamon Press, 1961. Price, $12.50.

The overlapping of the physical and the biophysical sciences is emphasized in this series. Volume 11 is comprised of six papers, with an added section of short discussions covering various phases of cytoplasmic particles and their role in protein synthesis. The six major papers are: "The Natural Radioactivity of the Human Body"; "The Engineering Approach to the Problem of Neural Organization"; "Effects of X-Rays on Nucleic Acid Biosynthesis and on the Activity of Nucleases in Mammalian Cells"; "The Effect of Ionizing Radiations and Tumour-Chemotherapeutic Agents on the Bone Marrow"; "Some Factors Influencing the Dispersion of Indicator Substances in the Mammalian Circulation"; and "The Buoyancy of Fish and Cephalopods." Excellent bibliographies, including 1960 references, accompany not only the six papers, but also the seventh section on cytoplasmic particles.

ENERGY PRINCIPLES IN APPLIED STATICS, by T. M. Charlton. 106 pages, diagrams, 6×9 in. London, Blackie & Son Ltd., 1959. Price, 25s.

The author has written this text in an effort to clarify the difference between the strain-energy and potential-energy methods applicable to the analysis of statical, plane pin-jointed systems. The author clearly demonstrates the principle of virtual work, and points out that Castigliano's "principle of least work" (which is a misnomer) is only one of two possible approaches to the analysis of linear statically-indeterminate structures, the other being the method of potential energy. The author devotes two (of a total of seven) chapters to the complementary-energy method of analysis. Our readers may recall Charlton's paper in this JOURNAL (Vol. 250, p. 543) on the analysis of redundant systems through the conception of conservation of energy.

DISPERSION RELATIONS & ELEMENTARY PARTICLES, edited by Cécile De Witt and R. Omnes. 671 pages, diagrams, $6\frac{3}{4} \times 9\frac{1}{2}$ in. Paris, Hermann & Cie., 1961. Distributed in U. S. by John Wiley & Sons, Inc. Price, $20.00.

This collection of eight treatises on various aspects of dispersion relations and elementary particles represents material presented at the 1960 summer session of the School of Theoretical Physics at the University of Grenoble. Two of the papers are in French, the others in English.

M. L. Goldberger's "Introduction to the Theory and Applications of Dispersion Relations" covers the historical background, two-dimensional representation, non-relativistic and relativistic theory, and applications to pion-nucleon scattering, forward scattering and non-forward scattering.

A. S. Wightman's two papers (one in English and one in French) deal with invariance in relativistic quantum mechanics (Lorentz and Poincaré groups, physically and not physically realizable states, inversions, infinite operators) and with analytic functions of several complex variables (holomorphy domains, continuity theorem, and others).

R. Omnes' contribution (in French) covers general properties of the S matrix reduction formulas, Dyson's formula, and holomorphy domains, with mathematical proofs and examples.

A. O. G. Källen discusses the properties of vacuum expectation values of field operators, covering field and particle concepts, reduction formulas, 2-, 3-, 4- and n-point functions.

G. F. Chew's contribution, "Double Dispersion Relations and Unitarity as the Basis for a Dynamical Theory of Strong Interactions," deals, for example, with the Lorentz-invariant amplitude, the Mandelstam representation, charge and spin, partial-wave amplitudes, pion-pion scattering, and the nucleon electromagnetic structure.

S. B. Treiman discusses weak interactions, including Leptonic and non-Leptonic reactions, and universal models. Included are

beta decay, μ-meson decay, decay of neutral K particles, and others.

The final contribution, by Y. Yamaguchi, deals with strong interactions of strange particles, covering isospin, spin-parity, K-N collisions, the Stark effect, and the application of dispersion theory to pology.

PRINCIPLES OF INTERNAL NAVIGATION, by C. J. Savant, Jr., R. C. Howard, C. B. Solloway and C. A. Savant. 254 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $9.75.

This first book devoted entirely to this rapidly expanding field covers the basic concepts of automatic navigation, the design and testing of autonavigation components, and the necessary mathematical analysis techniques. Recent advances are included, such as ballistic guidance and aircraft guidance. The text, which includes most of the unclassified material available, was originally prepared as a series of lectures for graduate students, but it is also a valuable reference for those working in the field. Among the 47 references (dated 1954–1959) is the O'Donnell paper appearing in this JOURNAL for 1958.

ZAHLENWERTE UND FUNKTIONEN AUS PHYSIK, CHEMIE UND ASTRONOMIE, GEOPHYSIK UND TECHNIK. II BAND: EIGENSCHAFTEN DER MATERIE IN IHREN AGGREGATZUSTÄNDEN. 7 TEIL: ELEKTRISCHE EIGENSCHAFTEN II (ELEKTROCHEMISCHE SYSTEME), by R. Appel, K. Cruse, P. Drossbach, H. Falkenhagen, G. G. Grau, G. Kelbg, E. Schmutzer and H. Strehlow, edited by Karl-Heinz Hellwege, Anne Marie Hellwege, Klaus Schäffer and Ellen Lax. Sixth edition, 959 pages, diagrams, 7¼ × 10¼ in. Berlin, Springer-Verlag, 1960. Price, DM 478.

This second section of Part 7, which has the general title of Electrical Properties, covers electrochemical systems in the careful and detailed manner with which all the volumes in this important series of tables and functions have been prepared. This sixth edition is divided into three main parts—Electrical Conductivity in Electrochemical Systems, Electromotive Forces, and Equilibrium in Electrochemical Systems. The first part covers tables of molten salts, pure liquids, aqueous and nonaqueous solutions and electrophoretic and electrokinetic potentials. The second part contains tables of reversible and irreversible potentials. The third part covers dissociation constants, acids, bases and indicators, and pH numbers with buffer solutions. The table of contents is clearly arranged so that a particular table may be quickly located. Literature sections follow all of the tables. The book is well printed and well bound, to withstand the hard usage demanded of a reference work of this sort.

THE PLASMA STATE, by E. J. Hellund. 190 pages, diagrams, 6 × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $6.50.

This book is an initial effort in presenting the fundamentals and significance of highly ionized gases, fully describing what is currently known of the plasma state in all its aspects.

The author defines plasma and explains the complexities of energy-exchange processes, after giving a brief historical outline of high-energy sources. Plasma morphology, energy exchange and hot gases are covered in separate chapters. A discussion of problems yet to be solved in regard to plasma is included in a chapter on the various kinds of plasmas, and there is an outline of devices, and information on the artificial and natural sources of high-energy particles. Industry and research workers will find a wealth of valuable information in this important new book.

TOPOLOGY, by John G. Hocking and Gail S. Young. 363 pages, diagrams, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1961. Price, $8.75.

Offering a unified and self-contained treatment of basic topology, this book is intended for a first course in the subject of two semesters in length, at the graduate level. The authors had a double goal in writing this text—to present the fundamentals of topology needed by every professional mathematician and also to erect a broad framework upon which the student of topology may build. As a result of this, almost every topic of interest in topology is mentioned.

The first half of the book is primarily set-theoretic and the second half is algebraic,

being particularly noteworthy for its detailed treatment of the elementary homotoph and homology theory of simplicial complexes. Some of the topics covered are the theory of continua, relative homology and cohomology, Peano continua, inverse limit sequence and the Eilenberg-Steelrod axioms for simplicial theory.

MODERN DIELECTRIC MATERIALS, edited by J. B. Birks. 246 pages, diagrams, plates, 6 × 9 in. New York, Academic Press, Inc., 1961. Price, $7.50.

This book should be of value to engineers, physicists and chemists who are interested in dielectric materials, whether in their development, manufacture or practical applications. Providing an introduction to the chief dielectric materials, there are chapters on paper, hydrocarbon insulating oils, chlorinated hydrocarbons, natural and synthetic rubbers, synthetic high polymers, silicones, ceramics, glass, mica and micanite, and fibres and textiles.

Since there are so many widely different materials involved, a team of specialists, each of whom is actively engaged in research and development in the field of which he writes, has been assembled under the editorship of Dr. J. B. Birks of the University of Manchester.

RADIOACTIVE WASTES, edited by John C. Collins. 231 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $8.00.

One of the most important contributions of recent years to the field of wastes engineering, this book covers the implications of radioactivity in regard to water supply, waste water disposal and disposal of radioactive wastes and radioactive gases. Of a fundamental nature, the first half of the book explains the problems which have to be understood when dealing with radioactive material. The second half deals specifically with the treatment and disposal of liquid, solid and gaseous radioactive wastes. This latter part also includes valuable information on the results of laboratory work and on the experience gained in operating treatment plants in different parts of the world.

ELECTRONIC MEASURING INSTRUMENTS, by Harold E. Soisson. 338 pages, diagrams, illustrations, 6 × 9 in. New York, Mcgraw-Hill Book Co., Inc., 1961. Price, $7.50.

This text will prove an excellent reference source for engineers for basic information on electronic measuring instruments. It provides an introduction to electronic measurement and control as used in the laboratory and industry, and offers a general background for advanced work.

The first six chapters acquaint the student with the theory and function of the component unit, covering the fundamental concepts needed to establish a common background for the laws, terms, basic concepts, symbols and nomenclature required for an understanding of the functions and use of electronic instruments. The last eight chapters show the components' function in an instrument system used to measure or control a laboratory or industrial variable. Some topics of special interest which are covered in the book are semiconductor diodes and transistors, radiation counting equipment and analog and decimal converter systems.

PROCEEDINGS OF THE SIXTH CONFERENCE ON MAGNETISM AND MAGNETIC MATERIALS, sponsored by the American Institute of Electrical Engineers and the American Institute of Physics. 395 pages, illustrations, 8 × 10½ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $10.00.

Issued as a supplement to the March 1961 *Journal of Applied Physics*, this volume contains the 149 papers presented at the November 1960 Conference on Magnetism and Magnetic Materials held in New York City. Fifteen topics (for example, Ordered Spin Systems, Metallic Films, Ferromagnetic Resonance, Anisotropy, Microwave Devices, etc.) are covered, with from five to sixteen papers in each topic. Publication of the proceedings within a very few months of the conference will enhance its value to those in the field.

TRANSISTOR LOGIC CIRCUITS, by Richard B. Hurley. 363 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $10.00.

The author covers both logical mathematics (including logical routines and blocks) and

the transistor circuits that implement them. Although developed as a text for seniors and graduate students, the book can be profitably used by practicing engineers. The book is roughly divided equally between switching theory and switching circuits. The end-of-the-chapter problems are designed to illustrate the text material rather than to challenge the student to creative thinking. Sample chapter headings are: Binary Arithmetic; Diode Switches and Logic Circuits; Minimization (both single function and multiple function); Sequential Logic—Counters and Registers; and Monostable and Astable Circuits.

HANDBOOK OF FLUID DYNAMICS, edited by Victor L. Streeter. 1240 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $24.00.

This is an engineering and scientific level handbook, prepared by a panel of more than 30 well-known engineers and scientists who represent a cross section of industry, government laboratories and universities.

Fluid-flow principles, theory, methods, and allied data are broadly covered. Although the coverage is highly practical with specific applications to the fields of hydraulic power, propulsion, aerodynamics, petroleum production, and chemical reaction, theory is included whenever necessary for a clearer understanding of the underlying principles. Divided into two parts, the first deals with fundamental concepts and the second is devoted to applied fields.

Many detailed illustrations are used as aids for the reader and to provide current values of important fluid dynamics data.

PLASTIC FLOW AND FRACTURE IN SOLIDS, by Tracy Y. Thomas. 262 pages, diagrams, 6 × 9 in. New York, Academic Press, Inc., 1961. Price, $8.50.

Volume 2 of the Mathematics in Science and Engineering series, this book will be of special use to engineers and students of the theory of plasticity in the solution of some of their problems. Based largely on material from articles published by the author in various mathematical journals during the last five years, this book is primarily a study of the propagation, growth, and decay of discontinuities in solids. The theory of characteristic surfaces and the spread of plastic into elastic regions is also included.

The author's objective discussion of the topics meets a proper mathematical standard.

MEASURE, LEBESQUE INTEGRALS, AND HILBERT SPACE, by A. N. Kolmogorov and S. V. Fomin, translated by Natascha A. Brunswick and Alan Jeffrey. 147 pages, 6 × 9 in. New York, Academic Press, Inc., 1961. Price, $4.00.

Originally published in Moscow in 1960, this is the second volume of the two-volume work *Elements of the Theory of Functions and Functional Analysis*.

The theory of measure and of the Lebesque integral, as presented in this text, is based on lectures given by A. N. Kolmogorov in the Mechanics-Mathematics Faculty of the Moscow State University. The final preparation of this book was carried out by S. V. Fomin. The book is divided into five chapters: Measure Theory, Measurable Functions, The Lebesque Integral, Functions Which Are Square Integrable, and The Abstract Hilbert Space: Integral Equations with a Symmetric Kernel.

RADIATION BIOPHYSICS, by Howard L. Andrews. 328 pages, diagrams, illustrations, 6 × 9 in. Englewood Cliffs (N. J.), Prentice-Hall, Inc., 1961. Price, $11.35 (trade); $8.50 (text).

This new work presents all significant information and pertinent biophysical techniques essential to an understanding of radiation biology, but never before brought together in a textbook. The author, Chief of the Radiation Physics Section of the National Cancer Institute, provides an important landmark in the study of a subject which has an impact on all facets of human existence. He presents the workable knowledge and mathematical background which are needed for treating mortality data, analyzing experimental results for target size and shape, and hitness number.

Including methods of dosage calculation not previously available in a textbook, working graphs, tables of constants and other data required in radiation computation, this book will be welcomed by those employed in the field of medical science, radiology, biology,

radiation biophysics, as well as all those concerned with the hazards and the rich promise of nuclear power.

TECHNIQUES OF HIGH ENERGY PHYSICS, Vol. 5, edited by David M. Ritson. No. 5 in the Interscience Monographs and Texts in Physics and Astronomy series. 540 pages, diagrams, illustrations, 6 × 9 in. New York, Interscience Publishers, Inc., 1961. Price $16.75.

This book was written, the editor states in his preface, to help bridge the gap "between the young physicist starting his research career and the professional who has grown within a given field." Among the eleven contributors to this work are men from IBM Corp., M.I.T., AVCO Research Laboratory, and Harvard, Princeton, Brown and Northeastern Universities.

The book is broken down into eleven general headings, which include Nuclear Emulsions, Scintillation and Cerenkov Counters, Beam Optics and General Properties of Particles and Radiation.

FREE RADICALS IN BIOLOGICAL SYSTEMS, edited by M. S. Blois, Jr., H. W. Brown, R. M. Lemmon, R. O. Lindblom and M. Weissbluth. 387 pages, diagrams, 6 × 9 in. New York, Academic Press, Inc., 1961. Price, $14.50.

Twenty-nine papers are included in these proceedings of a symposium held in March, 1960. Although the majority of the contributors are from the United States, there are papers from England, Germany and Sweden. Emphasizing the rapid rate of advancement in this field, the papers deal not with the question of *whether* free radicals occur in biological systems, but with specific mechanisms in which they *do* occur (for example, enzyme-substrate compounds, photodynamic systems, cytochrome, peptides, etc.).

SEQUENTIAL DECODING, by John M. Wozencraft and Barney Reiffen. 67 pages, diagrams, $5\frac{3}{4} \times 9$ in. New York, John Wiley & Sons, Inc.; Cambridge, The Technology Press; 1961. Price, $3.75.

The six chapters in this monograph present a unified approach to the communications problem of coding and decoding, from a probabilistic point of view. Chapter 1 relates coding theory to data communication. Chapter 2, on block codes, surveys prior results essential to the sequel. Sequential decoding and convolutional encoding are discussed in Chapters 3 and 4. Experimental results and modifications of the decoding procedure are contained in Chapter 5 on "Simulation," and the final chapter considers applications of the concepts to channel models and possible engineering applications. This is not a text, but a research monograph.

TRANSMISSION OF INFORMATION: A STATISTICAL THEORY OF COMMUNICATIONS, by Robert M. Fano. 340 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc.; Cambridge, The Technology Press; 1961. Price, $7.50.

Specifically directed to graduate students and engineers interested in electrical communications, this book is the most up-to-date one available in the field of information theory. The nine chapters cover transmission of information, a measure of information, simple message ensembles, discrete stochastic sources, transmission channels, channel encoding and decoding, encoding for binary symmetric channels, multinomials distributions, and encoding for discrete, constant channels. Appendix A contains problems for all the chapters; Appendix B presents 15 pages of entropy tables; and Appendix C is a table of the Gaussian distribution function.

PUBLICATIONS RECEIVED

INDUSTRIAL FURNACES, VOLUME 1, by W. Trinks and Matthew Mawhinney. Fifth edition, revised, 486 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $17.00.

TOOL STEEL SIMPLIFIED, by Frank R. Palmer and George V. Luerssen. Third edition, completely revised, 595 pages, illustrations, 6 × 9 in. Reading (Penna.), The Carpenter Steel Company, 1960. Price, $2.50 (postpaid in U. S.).

STRUCTURAL ENGINEERING FOR PROFESSIONAL ENGINEERS' EXAMINATIONS, by Max Kurtz. 341 pages, diagrams, 5½ × 8 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $9.00.

MARINAS, by Charles A. Chaney. Second edition, 247 pages, illustrations, 8½ × 11 in. New York, National Association of Engine and Boat Manufacturers, Inc., 1961. No price.

INTRODUCTION TO ENGINEERING MECHANICS, by John V. Huddleston. 474 pages, diagrams, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., 1961. Price, $9.75.

THE IMPACT OF THE NEW PHYSICS, by Frank Hinman. 168 pages, diagrams, 5½ × 8½ in. New York, Philosophical Library, Inc., 1961. Price, $4.50.

WATER TREATMENT FOR INDUSTRIAL AND OTHER USES, by Eskel Nordell. Second edition, revised, 598 pages, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $12.00.

PILE FOUNDATIONS, by Robert D. Chellis. Second edition, revised, 704 pages, illustrations, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $16.00.

TECHNICAL REPORT WRITING, by Fred H. Rhodes. Second edition, revised, 168 pages, 5½ × 8½ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $5.50.

A FIRST COURSE IN MATHEMATICAL STATISTICS, by C. E. Weatherburn. Second edition reprinted with corrections, 277 pages, 5½ × 8½ in. New York, Cambridge University Press, 1961. Price, $2.75 (paper).

THE METHODS OF PLANE PROJECTIVE GEOMETRY, by E. A. Maxwell. Reprinted, 230 pages, 5½ × 8½ in. New York, Cambridge University Press, 1961. Price, $1.95 (paper).

RADIO CONTROL MANUAL, by Edward L. Safford, Jr. 192 pages, illustrations, 5½ × 8½ in. New York, Gernsback Library, Inc., 1961. Price, $3.20 (paper).

Television Weather Satellite May Spot Hurricanes.—A new chapter in one of the nation's most successful space programs opened with the launching of the third Tiros "weather eye" satellite to observe weather patterns and changes around the world—and especially to report on the origin and development of hurricanes in the Atlantic and Caribbean areas.

The newest weather satellite was built for NASA's Goddard Space Flight Center by the same Radio Corporation of America engineering team that provided the Tiros I and II satellites and their associated ground systems. Where the first two models carried one wide-angle television camera for large-area viewing and one narrow-angle unit for detailed cloud study, the new Tiros carries two wide-angle cameras. It also differs from its predecessors in carrying a new experimental infrared sensing system developed by Professor Vernon Suomi, of the University of Wisconsin, to measure the heat radiation coming earthward from the sun, and the amounts absorbed and reflected by the earth and the atmosphere.

New Features on Tiros III

Identical in size and shape to the first two Tiros satellites, the 285-lb. Tiros III differs in only a few details from the earlier types. In addition to the substitution of a second wide-angle camera for the earlier narrow-angle type, and the addition of the new infrared experiment, it carries improved remote control programmers for operating the cameras and other electronic equipment in the satellite.

Each of the two cameras will photograph cloud patterns over an area about 700 miles on a side for transmission to the ground stations. Except for the replacement of the previous narrow-angle lens by a wide-angle type, the duplicate television camera, tape recording and transmission systems in the new satellite are identical to those in the earlier Tiros models.

The new infrared equipment carried for the first time on Tiros III consists of an omnidirectional sensor system carried on outriggers extending from opposite sides of the satellite's circular base. The sensors will determine the thermal radiation characteristics of the earth by measuring the heat radiated toward earth by the sun, and the amount that is re-radiated from the earth's surface and atmosphere. In addition, Tiros III carried other NASA infrared sensing equipment identical to that employed for the first time in Tiros II to measure heat variations on the earth's surface and in the atmosphere.

A less obvious change in Tiros III is the use of new transistorized circuits in the two electronic "clocks" which trigger the operation of the television cameras and other equipment at any desired point in orbit, in accordance with the program that is sent to the satellite as it passes within range of a ground station. The new circuits, replacing an arrangement of moving parts in the earlier Tiros clocks, are expected to add further to the demonstrated reliability of the satellite system.

Two significant changes also have been made in the RCA-operated ground station complex for Tiros III.

One of the two main ground stations, formerly at Fort Monmouth, N. J., has been moved to the NASA Wallops Station at Wallops Island, Virginia. The second main station remains at the Pacific Missile Range in southern California. Both are backed up, as before, by an additional ground station at the RCA Space Center, Princeton, N. J.

The new satellite is scheduled to operate for at least three month in an orbit similar to those of Tiros I and Tiros II—about 450 miles above the earth, in a path extending north to the latitude of mid-Canada, and south to the latitude of lower Argentina. Within this belt, the satellite will pass repeatedly over all parts of the earth in successive orbits, televising cloud formations and measuring infrared radiation.

Quick-Disconnect Coupling for Liquid Oxygen and Nitrogen.—Cryogenics, Inc., of Stafford, Va., has announced the development of a quick-disconnect coupling for use with liquid oxygen and nitrogen. The device fills the need for a coupling which is simple to connect and disconnect, and is ruggedized for missile and ground handling.

The oxygen and nitrogen couplings are keyed differently so that the possibility of erroneous connection is eliminated. The coupling is manufactured from standard hex stock so that no special tools are required for assembly.

The coupling consists of a male and female tapered connector which, when joined, form a leak-proof seal. The heart of the coupling is a yoke-type lever. Utilizing a leverage action with a high mechanical advantage, the coupling can be disengaged instantly with a minimum of force. The lever pivots on the female coupling and is fitted with slots that engage pins on the male assembly. The slots are engineered such that by depressing the lever, the male coupling is brought into position and firmly locked. In addition, the lever and the female connector are engineered to provide a safety lock in the form of a ball detent arrangement. This safety feature acts to prevent accidental movement of the lever.

In the locking operation, the male coupling firmly seats against two floating O-Rings within the female assembly creating a positive leak-tight seal.

By raising the lever, a cam action is produced which disengages the pins, resulting in a positive disconnect action even in the face of frost buildup.

The design of this coupling is such that icing of outside surfaces is minimized. The unit is constructed of a low thermal conductivity stainless steel. The two O-Rings which act as seals minimize heat transfer in that they permit a vapor buildup between the inside and outside surfaces and thereby provide a poor thermal conductive path.

The inner diameters of the male and female assemblies are identical and there are no obstructions in the flow stream, thereby, eliminating pressure drops. Couplings with inner diameters as small as $\frac{1}{2}$ in. and as large as 8 in. are available.

The coupling is suitable for remote operation and can also be provided with integral check valves to prevent fluid flow during disconnect operations.

Low Temperature Transformer.—A new kind of transformer that may become a major innovation in power technology was described by its inventor, Dr. Richard McFee, of Syracuse University and Arthur D. Little, Inc., at a meeting of the American Institute of Electrical Engineers in

Ithaca, New York. The new transformer was developed by Dr. McFee at A. D. Little, where it has been successfully operated at a level of 15 kilowatts.

According to McFee, transformers using the new design could be considerably lighter and less costly than conventional transformers. And they would not be subject to the substantial power losses caused by electrical heating, or to the temperature limitations of present-day electrical insulation.

The new design makes use of superconductivity—the ability of certain metals to conduct an electric current without any resistance at temperatures near absolute zero (−459.6°F.). Superconducting transformers have been considered before, but the fact that a moderately strong magnetic field can quench the superconducting state by restoring electrical resistance presented an apparently insurmountable obstacle. When an electric current flows through a coil, such as the windings of a transformer, powerful magnetic fields are generated about the windings and quench the superconductivity when they reach a critical field strength.

Dr. McFee discovered that interleaving layers of the primary and secondary windings would keep the total magnetic field strength below the critical level. Since the current in adjacent layers flows in opposite directions, the magnetic fields generated about these layers are also opposing. In effect, the individual magnetic fields nearly cancel one another and eliminate the problem of magnetic quenching.

The Gifford-McMahon helium refrigerator invented at A. D. Little provided a solution to another major problem in the development of a practical superconducting transformer. There had not been a simple, reliable method of maintaining the superconducting coils at the necessary low temperatures for an extended period. Now in its final stages of development, the new refrigerator will produce temperatures to −452°F., and will maintain them unattended for months of continuous operation.

Since the cost of refrigerating the whole transformer would be excessive, only the superconducting coils themselves are cooled, while the cores are kept at room temperature. Heat leaks into the system are controlled by vacuum insulation, liquid-nitrogen heat shields, and an ingenious arrangement of the superconducting input leads.

Superconducting transformers could eliminate a number of major problems that face designers of conventional transformers. The lack of electrical resistance in superconducting metals means that little heat is produced. Expensive devices for dissipating the heat generated in conventional transformers are therefore unnecessary in the new development. For the same reason, present power losses from heating in the windings would no longer exist, and engineers would not have to consider the deteriorating effect of heat on insulation.

The problem of excessive voltage loads can be handled in the same manner as in conventional transformers. Excessive current loads, however, require more elaborate safeguards in the new device because they could quench superconductivity and put the transformer out of service for a minute or more. Such outages can be prevented through the use of very fast-acting circuit breakers or external inductance coils.

Tiny Experimental Thin-Film Transistor.—An ultraminiature experimental transistor, so small that as many as 20,000 can fit on a postage stamp,

has been announced by the Radio Corporation of America. The transistor is made by depositing thin films by evaporation on an insulating base and is capable of shrinking the basic circuitry of a computer to the size of a book page. The basic circuitry of present-day computers ranges in size from the equivalent of a large hat box to a walk-in clothes closet.

Dr. James Hillier, Vice President, RCA Laboratories, described the development as "an important breakthrough in basic electronic circuitry that may well have a revolutionary impact upon the electronics industry."

According to RCA scientists, the new device, which has been tested successfully at RCA Laboratories, may open the way to new ultraminiature mass-production transistor circuits for many applications, especially in electronic computers and perhaps ultimately in other equipment such as thin-screen wall-type television receivers.

The active material used in the transistor is cadmium sulfide, a compound with considerably greater insulating properties than the germanium, silicon, and other semiconductor materials used in standard transistors.

In making the thin-film transistors, an evaporation process is used to deposit successive thin layers of cadmium sulfide and metal on a glass plate, creating a device that is only a few ten-thousandths of an inch thick. In the evaporation process, the cadmium sulfide crystals and the metal are heated in successive steps in a vacuum, turning to vapor that is collected by condensation on the glass plate in the same manner as steam condenses in a film on a cooler surface held over boiling water.

By using a special mask to cover portions of the plate during the process, the metal layers are deposited in a pattern that forms the electrical contacts needed to operate the transistor. The masking process also can be used to produce various patterns of connections among many transistors to complete a desired circuit at the same time that the transistors themselves are being made.

The completed transistor is not only very tiny, but it also incorporates an important operating feature not now used in commercial transistors. In conventional transistors having comparable functions, electrons flow more or less freely through the semiconductor material between two of the contacts, and the third element provides control by reducing the flow in varying degrees. The operating principle of the experimental thin-film transistor is exactly opposite. The insulating properties of the cadmium sulfide hamper the flow of electrons between two electrodes, and the third element provides control by increasing the flow in varying degrees.

Condensation Nuclei Detector.—General Electric Company has developed a device which promises important help in the battle against air pollution. It is so sensitive for detecting tiny airborne particles that it could find a single speck lost among 1000 trillion other specks. This would be like spotting a solitary grain of white sand on a beach full of yellow grains.

This "condensation nuclei detector" also has important industrial applications, although it is not yet commercially available. "Condensation nuclei" are the smallest airborne particles in existence. Invisible but extremely numerous, they are considered a major factor in smog formation.

The detector could be used to help forecast weather conditions by spotting nuclei present in our atmosphere from places hundreds or even thou-

sands of miles away. The nuclei would move as the air mass moved. With the use of auxiliary equipment, radioactive particles could be detected and, in most cases, traced to their source. Monitoring wilderness areas for forest fires is another possible use. Another unusual application is predicting the useful life of certain electrical equipment on the basis of a single operation. An example is that of the contact points in an electrical relay. Their life could be accurately predicted with the points coming together just once. There would be no visible signs of wear, even under a microscope.

This technique promises an enormous time saving, for it can accomplish in minutes or hours what normally takes months.

The detector works by causing droplets to be formed artificially around each tiny nucleus. The nuclei then grow to microscopic size. This is large enough to cause scattering of a light beam directed into their midst.

Measured electronically, the light scattering indicates the concentration of nuclei in the man-made fog. It takes but 5-thousandths of a second to form a measurable cloud.

Ability to detect the nuclei with a sensitivity of better than one part per 1000 trillion will prove a boon to all forms of air pollution work and should make possible the design of air filters with an effectiveness far exceeding any now in existence.

Monitoring air pollution, including the exhaust from automobiles, would be another important job for a nuclei detector of this high accuracy. It has been estimated that air pollution is presently costing the American public more than $10 billion a year in corrosion, crop damage and impairment of health.

Continuous operation of the device makes it of particular value in detecting and identifying gases. The gases cannot be detected directly, but an indirect measurement is possible if the gas molecules are combined with substances to give them a solid or liquid form. This is accomplished through the use of "converters" especially designed for the requirements of each basic group of gases.

The extreme sensitivity of the detector is due in large measure to its continuous operation, which also permits its use for automatic monitoring of the air in a remote location. The continuous operation is achieved by forming and measuring five to eight cloud masses a second inside the instrument's "cloud chamber." Previous laboratory models took intermittent measurements of the air, spaced more than a minute apart, and yielding results that had to be averaged out.

Micrometeorite Layer Discovered. —The Air Force has announced the discovery of what appears to be a dense band of micrometeorites which envelopes the earth at extreme altitudes.

This discovery was made recently by the Air Force Cambridge Research Laboratories, Laurence G. Hanscom Field, Bedford, Mass., when the contents of a novel rocket nosecone specially designed for trapping micrometeorites were examined. The nosecone was launched June 6 at White Sands, New Mexico. The density of the micrometeorites in this layer was totally unexpected. About 10 of these small particles struck each square centimeter of the detecting surfaces each second.

The dust particles, measured in microns (thousandths of a millimeter) apparently exist as a band about the earth that is independent of latitude

or longitude, Dr. R. K. Soberman, AFCRL project scientist, said. The precise altitude at which these micrometeorites are concentrated will be determined by future tests now planned by AFCRL.

The special nosecone for trapping and recovering these micrometeorites was designed by scientists and engineers of AFCRL and the Aerojet General Corp. It consisted of a pod-like arrangement made up of individual petals or leaves. At an altitude of 47 miles these leaves opened like a blossoming flower.

The nosecone was carried to a maximum altitude of 102 miles over the desert at White Sands, New Mexico. The leaves closed again when the rocket descended to 65 miles. Altogether the nosecone remained opened for about 4 minutes.

Two types of micrometeorite detectors were used. The first type consisted of a triple layer of physically separated Mylar and Plexiglas sheets. Each sheet measured about 2″ × 3″. Detectors of this type were located on each of the leaves of the nosecone. The top layer consisted of a Mylar-film ¼ mil (1/4000ths in.) thick. The second layer, also of Mylar was 1 mil. The bottom layer was a ⅛″ thick sheet of Plexiglas. The micrometeorites traveling at speeds up to 47 miles a second, passed through the 2 Mylar-film layers, creating many holes. Most of these were microscopic in size, although many were visible to the naked eye. When they struck the harder Plexiglas layer below, small craters, also readily visible, were created.

In addition to detectors of this type, relatively thick films of 3 plastics of varying hardness—Millipore, Formvar, Lucite—were used. Craters similar to those formed in the Plexiglas were detected in the Lucite film. Most of the evidence of micrometeorite contact, however, was apparent only through microscopic examination.

Few micrometeorites themselves were collected since, for the most part, they vaporized on contact with the detecting surfaces. Some of these extra-terrestrial particles, however, were collected intact. Of particular interest to AFCRL scientists is the residual bits of meteorite material lining the walls of the craters. These residual materials will be analyzed by physical, chemical and radioactive techniques.

Dr. Soberman suggested that the micrometeorite layer is probably formed by electrostatic trapping. Complete analysis, Dr. Soberman said, may take several years. The analysis will be done for the most part by electron microscopy techniques. The electron microscope must be used because many of the individual particles to be examined are smaller than the wave length of light. Complete examination of the 1 square inch of material, for example, may take a single scientist one full year. A large number of samples are available and AFCRL will furnish these to interested and qualified scientists.

One limitation on the present study is that there is no way of knowing with precision the altitude at which the meteorite layer is concentrated.

AFCRL hopes to determine this when a specially instrumented rocket is launched from Eglin AFB, Fla., in the near future. Improved microphone detection techniques which record the impact of these tiny particles on special metallic surfaces will be used in this test.

Instrumentation Rack for Sounding Rockets.—The Geophysics Corporation of America, Bedford, Mass., has developed a standardized instrumen-

tation rack for upper atmosphere sounding rockets. The rack is designed to accommodate equipment payloads of variable size and configuration and is able to withstand severe conditions of shock and vibration.

Built to package instrumentation payloads for rockets launched both in the United States and in foreign countries, the new rack is made of 6061-T6 aluminum with stainless steel hardware. It has four struts, one of which serves as a wiring channel and contains a series of spaced openings for plug-in assemblies. Although its length may be varied according to the needs of each experiment, the model available for ASP, CAJUN or APACHE rockets stands $25\frac{1}{16}$ inches high and weighs 8.5 lb. empty.

Evidence of the new rack's sturdiness was demonstrated recently in a rocket launching in which the rocket blew up 25.6 sec. after launch. Complete telemetry recordings continued to be received from the instrumentation aboard until the equipment impacted approximately 297.4 sec. later. From the telemetry records it was calculated that the rack and equipment was subjected to 1,000 "G" forces of radial acceleration, and to an unmeasured but extremely high amount of vibration

Multiple Diode.—The Radio Corporation of America recently introduced the electronic industry's first multiple germanium switching diode in a new bid to simplify the manufacture of computers. These new multiple diodes consist of two and three semiconductor diodes of advanced technology, packaged as single, integrated units, according to Dr. Alan M. Glover, Vice President and General Manager, RCA Semiconductor & Materials Division.

He said they are expected to simplify the assembly of computers by reducing the number of separate components and electronic connections required to build them.

Dr. Glover explained that diodes are small, individual, two-terminal devices used to help translate into computer electronics the principle that every question, no matter how complex, can be answered by reducing it to a series of "yeses" and "noes."

In conjunction with transistors, he said, diodes perform these "yes" and "no" functions by opening or closing thousands of "logic gates" (circuits) that control the flow of coded electronic signals through the computer.

"Building this logic network can be a very complicated and time-consuming job," Dr. Glover said. "However, if individual diodes are replaced by multiple diode assemblies, the job can be greatly simplified while giving rise to such other potential benefits as quicker assembly schedules, greater space savings and lower production costs."

Commenting on the advanced technology embodied in the new diodes, Dr. Glover disclosed that they are manufactured by a new process, unique to the diode field, that provides very high switching speeds as well as great mechanical and electrical stability.

"The simplicity and reliability of this unique process have been matched with a simplicity and reliability of design which," he said, "gives our new germanium diodes wide utility with high performance and low cost. They should prove of exceptional value to the entire computer industry."

Automated Production of Deposited Carbon Resistors.——A wholly new concept in the automatic manufacture of electronic components that combines statistical quality control with

fully mechanized processing—all under the guidance of a digital computer—was given its first public demonstration recently at the North Carolina Works of the Western Electric Company.

The block-long array of interconnected robots, observed in operation by newspaper reporters and technical editors, manufactures deposited carbon resistors. The resistors are electronic components that must meet severe electrical requirements, yet operate thousands of years on the average without failure. So great is the reliability demanded of the product that manual techniques of manufacture are impractical. The resistors are used in immense quantities and in a variety of electronic equipment including radar and missile control systems.

In addition to controlling step-by-step details of manufacture, the computer can accept a production schedule for a full month's output and thereafter automatically issue detailed instructions to the fabricating machines to make and package any or all of four basic power sizes of resistors in such quantities and resistance values as may be desired.

According to engineers who developed the automated line, it is "the first time that a computer has been used in conjunction with a statistical quality control system to control a series of fabricating machines which produce precise, electronic components."

The resistors look like tiny black firecrackers with copper wire "fuses" extending from the ends. Each consists of a short ceramic rod or core which is first coated with carbon. A conductive gold coating is applied to each end of the core for attaching a cap and wire terminal. Then a spiralled groove is cut into the carbon film, barber-pole fashion, to change the electrical path on the core and raise the resistance to the desired level. Finally, the resistor is encased in a cylindrical, epoxy plastic cover for protection. The machines work on a three second cycle, producing a resistor during each cycle. Maximum production rate is 1200 finished resistors per hour.

There are eleven machine stations on the line, in addition to the computer and control cabinets. Three of these stations are used only for testing. Length of the line is 110 feet, with a maximum machine width of about six feet.

To the observer's eye, the line's eleven machine stations have the trim, console look characteristic of modern industrial machines. One, in particular, puts on a brilliant show. Known as the "sputtering machine," it sprays the ends of the resistors with microscopic particles of pure gold inside gas-filled bell jars. The discharge of high-voltage electricity that propels the precious metal creates a veritable aurora borealis in miniature of flickering reds and purples.

In missile systems, deposited carbon resistors are required by the millions and they must work under extreme conditions of cold, heat, humidity, vibration and shock. In certain defense equipment, a failure rate of no more than one per 200 million hours (some 23,000 years) of operation is permissible.

It is mainly this reliability requirement which triggered the design and development of the machines shown here today. Until now, individual, precision products have been made by manual or semi-automatic processes that invited contamination from handling and related shortcomings of human control.

By adopting advanced automation, contamination disappears as a problem, production is vastly increased and, most vital, a level of reliability is attained that manual methods cannot match, advantages that are reflected in low unit costs.

Western Electric will continue the use of manual methods for the manufacture of conventional resistors.

YOU CAN ADVANCE SCIENCE EDUCATION

Today, more than ever before in its 138-year history, there is vital need for The Franklin Institute effectively to promote education in science and technology. It is imperative that we meet this challenge by providing adequate educational opportunities in these fields. This requires vision, objective planning, and money. We have more than enough of the first two requisites, but far too little of the third.

Our programs are aimed at professional scientists and industry, as well as the lay public and young people seeking inspiration and guidance in choosing a career. The Institute's educational programs are impressive. They begin with students in the early grades of our elementary schools and continue throughout an individual's professional or industrial life. With more funds at our disposal, the scope and vigor of these activities could be greatly increased and increasingly effective.

The Franklin Institute is not richly endowed. It is a non-profit organization, depending for encouragement and support on an understanding public. Capable and conservative management assures wise administration of all funds.

Your gift or bequest, large or small, will be deeply appreciated and will be used effectively to broaden the Institute's educational usefulness. There is a warm satisfaction in giving financial support to an organization that has pioneered in, and is dedicated to, the advancement of science and technology.

When property is transferred, title should be in the name of The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts.

The Secretary of The Franklin Institute will gladly furnish you with additional information. Write to him at The Franklin Institute, Benjamin Franklin Parkway at Twentieth Street, Philadelphia 3, Pennsylvania.

If you wish to give securities: (1) Send the stock certificates to The Franklin Institute, Philadelphia 3, Pennsylvania, and leave the assignment space blank. (2) Execute a stock power and mail this to the Institute *separately*, leaving the assignment space blank. (The Institute, upon request, will gladly furnish blank stock power forms.)

STAFF POSITIONS

. . . are available for qualified scientists and engineers in the Laboratories of The Franklin Institute. These Laboratories comprise one of the country's finest independent research organizations, and provide R&D services to both industry and government, in a variety of fields:

chemistry, solid state sciences, applied physics, operations analysis, engineering psychology, and chemical, electrical, mechanical and nuclear engineering.

There are openings, in groups working on challenging problems in both research and development, for men with proven abilities in any of these fields.

THE FRANKLIN INSTITUTE LABORATORIES
FOR RESEARCH AND DEVELOPMENT

20th Street and Benjamin Franklin Parkway
Philadelphia 3, Pennsylvania . . . LOcust 4-3600

Send complete resumé to
Mr. John E. Christ, *Director of Personnel*

Or write for a brochure describing activities of the Laboratories

BUSINESS
MATH
ENGINEERING

Now in the fourth year of operation, The Franklin Institute Computing Center has shared its technical know-how, modern electronic equipment and experienced personnel with hundreds of progressive industries and government agencies across the nation.

We offer the services of creative people, highly skilled in their respective fields and ably trained in the application of these skills to the ever expanding area of electronic computers and data processing systems.

This staff is now available for analysis, system design, programming or coding of projects of unlimited scope or context. Input to our large scale computer and peripheral equipment is acceptable in any form. Results are provided on cards, plastic or metallic tape, and in completely edited printed copy. Our extensive library of automatic coding, engineering, data processing and mathematical routines is available to all users, and machine time is provided with or without the services of programming personnel.

THE FRANKLIN INSTITUTE
Computing Center

20th Street and Benjamin Franklin Parkway
Philadelphia 3, Pa. LOcust 4-3600